A SET OF GENERALIZED FUNCTIONS AND ITS APPLICATION TO ELECTRICAL SYSTEMS

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SUMMARY

The impulse, or delta function, $\delta(t)$ is an important tool in applied mathematics since it simplifies the derivation of many results that would involve complicated arguments otherwise. The characterization of the concept of the impulse function is difficult, and its successful use in engineering has depended more on the insight of the user than on the utilization of some rigorous mathematical system.

Any definition of the impulse function is inadequate in a rigorous mathematical sense if it is viewed as a normal function. The difficulties involved in defining the impulse function can be eliminated if it is introduced as something more general than a normal function.

A number of theories for the impulse function have been developed. in most cases, these theories do not lead to mathematical systems in which consistent definitions of such operations as differentiation, integration, multiplication and addition are defined for normal functions as well as all impulse, and other improper functions, used in engineering and physics.

In the theory of distributions, a unified approach is given for improper functions such as the impulse δ and its derivatives δ^k, where $k = 1,2,3,\ldots$. The difficulties involved with these improper functions are eliminated by recognizing that they are not normal functions with definite values for every value of an independent variable, but that these improper functions are new concepts specified by their properties.

Another important unified approach to improper functions such as δ and δ^k is given in Mikusinski's theory of convolution quotients. In this theory as in the theory of distributions, the concept of a function must be generalized in order to develop a rigorous mathematical system that contains impulsive type functions such as $\delta(t)$.

It can be said that the methods for obtaining generalized mathematical systems have been highly developed by mathematicians. However, these methods have rarely been used in engineering analysis. This seems to be the case because of either or both of the following reasons. The techniques make use of mathematical principles that are at present unfamiliar to most engineers. It is also difficult to develop intuitive meaning for the new concepts.

It is likely that most electrical engineers would have great difficulty developing a physically meaningful insight into the behavior of a system of differential equations excited by distributions.

The purpose of this research is to develop a satisfactory generalized mathematical system in which impulse type functions are characterized and in which operations on these functions are defined in a manner that is compatible with the electrical engineers intuitive viewpoint of the concept.

The operations of integration, differentiation, addition, and a form of multiplication are to be defined in the mathematical system developed. It is intended that these generalized operations, which are used to manipulate generalized functions such as the impulse, be intimately related to ordinary mathematical operations used in engineering to manipulate normal functions.

Hopefully the mathematical system developed can be of more value to the electrical engineer than are such mathematical systems as distribution theory wherein the characterization of generalized functions is by something other than normal functions, and where the operations defined in the system have little in common with ordinary mathematical operations familiar to most engineers.

The motivation behind the generalized mathematical system developed in this work is the possibility of defining generalized functions to be families of sequences of normal functions. The families of sequences constituting the generalized functions are considered to be disjoint families. That is, any sequence in one family, and hence in one generalized function, is not contained in any other family of sequences of normal functions. For this situation, a sequence of normal functions that represents some generalized function cannot possibly represent any other generalized function.

A normal function $f(t)$ can be called a representation of a generalized function g if the <u>constant</u> sequence $\{f_n(t)\}$, where $f_n(t) = f(t)$ for each $i = 1,2,\ldots,$ is a sequence in the family of sequences defining the generalized function g. It is possible to have generalized functions that are not represented by any normal functions. The δ-function and its derivatives δ^k are found to be families of sequences of normal functions with this property.

It is proposed that a normal operation, F, such as integration or differentiation, familiar for normal functions, be extended to the generalized functions in the following manner. For an arbitrary generalized function g_1, any sequence $\{b_n\}$ in g_1 is considered. Using the ordinary operation F on the normal functions of the sequence $\{b_n\}$, the sequence of normal functions $\{F(b_n)\}$ is determined. If the family of sequences $\{F(b_n)\}$ determined for all $\{b_n\}$ in g_1 is contained in some unique generalized function, say g_2, the generalized operation F on g_1 is defined to be the generalized function g_2. The generalized operation F is defined for a set of generalized functions G if the preceding remarks hold for every generalized function g_t in G.

An operator f^k, $k = 0, \pm 1, \pm 2, \ldots$, related to ordinary integration and differentiation, is defined for the functions of set B. The operator f^k is used in grouping the families of sequences of normal functions composed from B into the generalized functions of set G. This is accomplished by using f^k to define an equivalence relation for the infinite sequences of normal functions that can be composed from the set B.

The set G of generalized functions obtained in this work contains the improper impulse type functions δ^k, $k = 0, 1, 2, \ldots$; and most of the normal functions used in electrical engineering are embedded in G. The consistent operations of addition, differentiation, integration, and multiplication by functions in C^∞ are defined in G in a meaningful way. The generalized operations defined in G are consistent in that they are equivalent to the corresponding ordinary operations on any normal function embedded in G whenever the ordinary operations are defined for the normal function.

In many engineering problems, an ordinary operation such as differentiation of a normal function like the step function $u(t - t_j)$ fails to be defined in the ordinary sense. By application of the generalized mathematical system presented here, such operations on normal functions may be well defined in the extended or generalized sense. In particular, it is found that the generalized derivative of the generalized function represented by $u(t - t_j)$ is the impulse function $\delta(t - t_j)$. Also, it is found that the step function $u(t - t_j)$ represents the generalized function in G that is the generalized integral of $\delta(x - t_j)$.

The behavior of many important electrical systems can be described by systems of linear constant coefficient differential equations (l.c.c.d.e.). The response of a linear electrical system to some excitation of interest may not be defined as a classical solution of the system of l.c.c.d.e.'s describing the electrical system. For example, the solution of a system of l.c.c.d.e.'s excited by impulse type functions can not be determined by the classical solution of the system of equations. However, it may be possible to determine the system response to "improper" excitations by obtaining the generalized solution of the system of differential equations describing the electrical system.

Methods for sòlving linear constant coefficient differential equations and systems of these equations in G have been developed in this work. In each case the solution in G is found to be unique in the same sense that the ordinary solution of a l.c.c.d.e. or system of l.c.c.d.e.'s is a unique solution when sufficient boundary conditions are applied. In particular, for the m boundary conditions $\xi_1,\ldots,\xi_m$ at time $t = a_1$ there exists a unique solution in G for the m^{th} order l.c.c.d.e. that is excited by a function such as δ^k in G.

Several important examples are given for the application of the generalized mathematical system to electrical circuits. The solutions of these problems using the generalized mathematical system developed in this work are found to be consistent with the solutions obtained by the classical theory of Laplace-transforms.

It is concluded that the generalized mathematical system developed in this work has considerable potential intuitive content for the electrical engineer. This conclusion follows since the generalized functions of the set G are composed directly from sequences of normal functions, and the generalized operations defined on G are intimately related to the corresponding ordinary operations on the normal functions of any sequence representing a generalized function in G.

CHAPTER I

INTRODUCTION

The impulse function, or Dirac δ-function, is an important tool in applied mathematics since it simplifies the derivation of many results that would involve complicated arguments otherwise. Characterization of the concept of the impulse function is difficult and its successful use in electrical engineering has depended more on the insight of the user than on the utilization of a rigorous mathematical system.

One practical use for the impulse function is found in linear circuit theory. Any set of arbitrary initial currents and charges in such a circuit may be replaced by an appropriate set of voltage and current impulse sources connected in the network. Superposition of their individually produced responses and that response due to some specific excitation, all computed for initial <u>rest</u> conditions, yields the desired net response.(1)

Another important application of the δ-function is in the characterization of a fixed parameter linear system by its impulse response. It can be shown that the response of an initially relaxed fixed-parameter-linear system to a unit impulse is equal to the inverse Laplace transform of the transfer function of the system.(2) The impulse response is then another way to characterize the input-output relation of the system. The knowledge of the impulse response for a linear-fixed-parameter system is of key importance in determining the realizability of the system, the stability of the system, and the response of the system to a general excitation.[3]

The impulse type functions are also found in the statistical theory of signals. For example, the power density spectrum of the unit step function is an impulse function.[4]

In most technical literature the delta function is defined in one of the following three ways:[5,6]

(i) By the equation

$$\int_{-\infty}^{\infty} \delta(t)dt, \quad \text{where} \quad \delta(t) = 0 \quad \text{for} \quad t \neq 0.$$

(ii) As the limit

$\delta(t) = \lim_{n \to \infty} f_n(t)$ of a sequence of normal functions satisfying

$$\int_{-\infty}^{\infty} f_n dt = 1, \quad \text{and} \quad \lim_{n \to \infty} f_n(t) = 0 \quad \text{for all} \quad t \neq 0.$$

(iii) By the "sifting" property

$$\int_{-\infty}^{\infty} \delta(t)\, f(t)dt = f(0), \quad \text{where} \quad f(t) \quad \text{is an arbitrary}$$

normal function that is continuous at $t = 0$.

Any definition of the δ-function is inadequate in a rigorous mathematical sense if the delta function is viewed as a normal function.* The difficulties in defining δ can be eliminated if δ is introduced as something more general than a normal function.

Background

A number of theories for the δ-function have been developed.[7,8]

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*A normal function $f(t)$ assigns one finite value for each value of the independent variable t, where t is a real number. Functions such as $\sin(\omega t)$ and e^{at} are normal functions when ω and a are real or complex numbers.

In most cases, these theories do not lead to mathematical systems in which consistent definitions of such operations as differentiation, integration, multiplication and addition are defined for normal functions as well as all impulse, and other improper functions, used in engineering and physics.[9]

In the theory of distributions, a unified approach is given for improper functions such as δ and δ^k. The difficulties involved with such improper functions are eliminated by recognizing that such functions are not normal functions with definite values for every value of an independent variable, but that the improper functions are new concepts specified by their properties.[10,11,12,13] A more detailed discussion of this important theory will be given later in the introduction.

An entirely different concept of generalized functions is put forth in Mikusinski's "convolution-quotients."[14,15] This concept is less general than that of distributions in that it is not designed to cope with functions of real variables whose ranges are unrestricted in the real line E_1. It is most successful in its application to functions of a single non-negative variable, although it has been extended to functions of a real variable ranging over a finite interval.

It can be said that the methods for generalizing function spaces have been highly developed by mathematicians. However, these methods have rarely been used in engineering analysis. This seems to be the case because of either or both of the following reasons. The techniques make use of mathematical principles that are at present unfamiliar to most engineers. It is difficult to develop intuitive meaning for the new concepts.

It is likely that most electrical engineers would have great difficulty developing a physically meaningful insight into the behavior of a system of differential equations excited by distributions.

Objective

The objective of the present research is to develop a satisfactory generalized mathematical system in which impulse type functions are characterized and in which operations on these functions are defined in a manner that is compatible with the electrical engineers intuitive viewpoint of the concept.

The operations of integration, differentiation, addition, and a form of multiplication are to be defined in the mathematical system developed. It is intended that these generalized operations, which are used to manipulate generalized functions such as the impulse, be intimately related to ordinary mathematical operations used in engineering to manipulate normal functions.

Hopefully the mathematical system developed here can be of more value to the electrical engineer than are such mathematical systems as distribution theory wherein the characterization of generalized functions is by something other than normal functions, and where the operations defined in the system have little in common with ordinary mathematical operations familiar to most engineers.

In most electrical engineering work the unit impulse is considered to be a narrow pulse of unit area and of unspecified shape. In general, the response of some system upon application of this impulse as an excitation is desired. An approximation of the unit impulse response is obtained by application of some narrow pulse of unit area. The impulse response is obtained by considering the limit of responses to a sequence of such pulses as the pulse width approaches zero. This viewpoint is useful since the response of many physical systems is substantially independent of pulse width and shape, but not independent of pulse area, so long as the duration of a pulse is sufficiently small.

In general, the sequence of system pulse excitations for which there corresponds a given limit of system responses is not a unique sequence of excitations. The limit of responses of a given system excited by an impulse type sequence is found to be the same for a family of sequences of pulse excitations and is not obtained from just one sequence of input pulses.

The foregoing discussion indicates that the concept of the limit of a sequence of functions might be assigned an important role in a mathematical system that characterizes impulse type functions as families of sequences of normal functions. This is possible in spite of the fact that the limit of a sequence of functions representing an impulse is not a representation of that impulse function.

Outline of Research

The motivation behind the generalized mathematical system developed in this work is the possibility of defining generalized functions to be families of sequences of normal functions.[16] The families of sequences constituting the generalized functions are considered to be disjoint families. That is, any sequence in one family, and hence in one generalized function, is not contained in any other family of sequences. For this situation, a sequence of normal functions that represents one generalized function cannot represent any other generalized function. A normal function $f(t)$ can be called a representation of a generalized function if the constant sequence $\{f_n(t)\}$, where each member $f_n(t)$ of the sequence is $f(t)$, is a sequence in the family of the generalized function. It is possible to have generalized functions that are not represented by any normal functions. The δ-function and its derivatives δ^k are found to be families of sequences of normal functions that have this property.

It is proposed that a normal operation, f, such as integration or differentiation, familiar for normal functions, be extended to the generalized functions in the following manner. Let g_1 be an arbitrary generalized function, let $\{b_n\}$ be any sequence in the family of sequences g_1, and let the sequence $\{f(b_n)\}$ corresponding to $\{b_n\}$ be constructed. If the family of sequences $\{f(b_n)\}$ determined for all $\{b_n\}$ in g_1 is contained in some unique generalized function, say g_2, the generalized operation of f on g_1 is the function g_2. The generalized operation f is defined for a set of generalized functions G if the preceding remarks hold for every generalized function g_t in G.

The above definition of a generalized operation on G is clarified in Figure 1. In the Figure, g_t is considered to be an arbitrary member of G. If the ordinary operation f on the normal functions in the sequences of g_t maps the sequences of g_t into a unique generalized function of G, the generalized operation f is defined on G and the result of the generalized operation f on g_t is the generalized function $f(g_t)$ in G.

The problems and results of the research will now be outlined

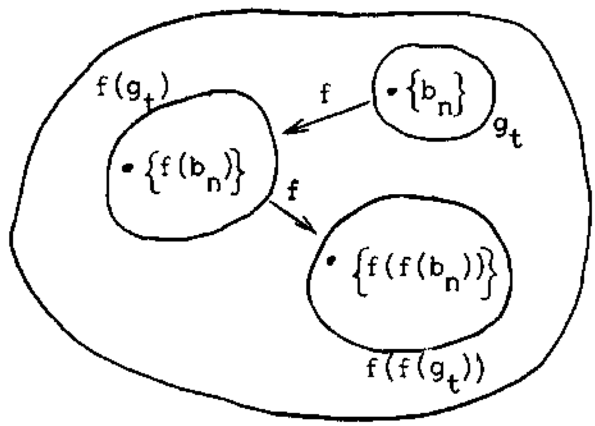

Figure 1. Generalized Operations on the Set of Functions G.

briefly. No detailed proofs of the results obtained are given in this outline; however, the discussion should be sufficient to explain the contents of the work.

The second chapter of this work is devoted to the development of a generalized mathematical system in which a set G of generalized functions is obtained. The generalized operations of integration, differentiation, addition, and multiplication by normal functions of the set C^{∞} are defined on G. The set C^{∞} is the set of those normal functions that are all-order differentiable everywhere on the real line E_1.

The development of the mathematical system begins with the selection of a set B of normal functions that are to be the members of the sequences in the generalized functions of G. The set B selected contains normal functions each of which is composed of a finite number of parts of functions in C^{∞} on each finite interval of E_1.

An operator f^k, related to ordinary integration and differentiation, is defined for functions of B. The operator f^k is used in grouping families of sequences into the generalized functions of set G. Here k assumes the values 0, ±1, ±2,... .

The set $\mathcal{B}$ is defined to be all infinite sequences of the form $\{b_n\}$ where b_n is a member of B for each n = 1, 2, 3,... . A reasonable approach to take in creating a set of generalized functions from the normal functions of B is to attempt to divide $\mathcal{B}$ into disjoint families of sequences and to call these families the generalized functions. However, a method has not been found for partitioning $\mathcal{B}$ in such a way that the delta function and its derivatives are members of G and such that the four generalized operations mentioned above are definable on G.

A successful development of G is possible using a subset of sequences S of the set $\mathcal{B}$. One property of set S is that each normal function in each sequence of S is a member of C^∞ on E_1. This condition on S is indeed restrictive, but it is found that S can be partitioned to obtain G where G contains the impulse function and all orders of its derivatives. Moreover, the four generalized operations mentioned before can be defined in a useful way on the set G obtained by partitioning S.

The operator f^k is used to define an equivalence relation for sequences in S. Two sequences $\{b_n\}$ and $\{b_n^*\}$ in S are defined to be equivalent sequences of S, and are written $\{b_n\} \sim \{b_n^*\}$ if the limits $\lim_{n\to\infty} f^k(b_n)$ and $\lim_{n\to\infty} f^k(b_n^*)$ exist and agree for all but possibly a finite number of points in any finite interval of E_1. As before, k is any integer 0, ±1, ±2,... . This equivalence relation partitions S into disjoint families of sequences since any two sequences are either equivalent or not equivalent.

For the set G obtained by partitioning S, the four generalized operations of differentiation, integration, addition, and multiplication by functions in C^{∞} are possible for functions of G as shown by the following arguments.

(i) If g_t is any member of G and if $\{b_n\}$ is any sequence in g_t, it is shown that the sequence $\{D_x^1 b_n\}$ is in some generalized function of G. Moreover, it is shown that the member of G containing $\{D_x^1 b_n\}$ is the same for any sequence in g_t. The generalized derivative of g_t in G is defined to be the unique member of G containing the sequence $\{D_x^1 b_n\}$ where $\{b_n\}$ is an arbitrary sequence in g_t.

(ii) If g_t is any member of G and if $\{b_n\}$ is any sequence in g_t, it is shown that the sequence $\left\{\int_{a_1}^{x} b_n\, dt\right\}$ is in some generalized function of G. Also, it is shown that the member of G that contains $\left\{\int_{a_1}^{x} b_n\, dt\right\}$ is the same for any sequence $\{b_n\}$ in g_t. In the integral $\int_{a_1}^{x} b_n\, dt$, a_1 is a fixed real constant. Of course it will be true that the sequence of derivatives obtained from $\left\{\int_{a_2}^{x} b_n\, dt\right\}$ will be $\{b_n\}$ even if $a_1 \neq a_2$.

The generalized integral of g_t in G is defined to be the unique member of G containing the sequence $\left\{\int_{a_1}^{x} b_n\, dt\right\}$, where $\{b_n\}$ is an arbitrary sequence of g_t.

(iii) If g_{t1} and g_{t2} are two arbitrary members in G and if $\{b_n\}$ is any sequence of g_{t1} while $\{b_n^*\}$ is any sequence of g_{t2}, it is shown that the sequence $\{b_n + b_n^*\}$ is in some generalized function of

G. Also, it can be shown that the member of G containing the sequence $\{b_n + b_n^*\}$ is the same for any pair of sequences $\{b_n\}$ and $\{b_n^*\}$ selected from g_{t1} and g_{t2} respectively. The generalized sum of g_{t1} and g_{t2} in G is defined to be the unique member of G containing the sequence $\{b_n + b_n^*\}$ where $\{b_n\}$ is <u>any</u> sequence g_{t1} and $\{b_n^*\}$ is <u>any</u> sequence of g_{t2}.

(iv) If g_t is any member of G and if $\{b_n\}$ is any sequence of g_t, it is shown that the sequence $\{g \circ b_n\}$ is in some generalized function of G for any given normal function $g(x)$ in C^∞. In addition, it is shown that the member of G containing the sequence $\{g \circ b_n\}$ is the same for any sequence $\{b_n\}$ in the generalized function g_t when $g(x) \varepsilon C^\infty$. For $g \varepsilon C^\infty$, the generalized product of g and g_t in G is defined to be the unique member of G containing the sequence $\{g \circ b_n\}$, where $\{b_n\}$ is an arbitrary sequence in g_t.

A normal function b in set B is defined to be embedded in the generalized functions G if the constant sequence $\{b_n\}$, $b_n = b$, is equivalent to some sequence in a generalized function of G. If the constant sequence $\{b_n\}$, $b_n = b$, is equivalent to a sequence in some generalized function g_t of G, then this constant sequence is equivalent to every sequence in the family g_t. However, the sequence $\{b_n\}$, $b_n = b$, cannot be equivalent to any sequence of any generalized function other than g_t. It follows then that an embedded function of B is a representative of a <u>unique</u> generalized function of G.

The four generalized operations derived in Chapter II are consistent with their ordinary counterparts in the following sense. If the normal function f of B is embedded in G at g_t and if $D_x^1 f$ is defined on

E_1, then $D_x^1 f$ is in B and is embedded in G at the generalized derivative of g_t. If $f \varepsilon B$ is embedded in G at g_t, the function $\int_{a_1}^{x} f\,dt$ is in B and is embedded in G at the generalized integral of g_t. If $g \varepsilon C^{\infty}$ and if $f \varepsilon B$ is embedded in G at g_t, the function $g \cdot f$ is in B and is embedded in G at the generalized product of g and g_t. Finally, if f and h are normal functions in B with f embedded in G at g_{t1} and h embedded in G at g_{t2}, the function $f + h$ is in B and is embedded in G at the generalized sum of g_{t1} and g_{t2}.

In Chapter III, some particular generalized functions in G are considered and the type of function embedded in G is described in more detail. The delta function is defined in G and is a unique generalized function. The derivatives of all orders for the delta function are also defined in G and each derivative is a unique generalized function. The derivatives of the delta function are found to be related by the property

$$D^1(\delta^m) = \delta^{m+1}$$

where, for each $m = 0, 1, \ldots;$ δ^m is the m^{th} generalized derivative of δ and where D^1 is the symbol for first order generalized differentiation defined on G. It is shown also that with minor exceptions, the set of normal functions B is embedded in G. That is, almost every function of B is embedded in G. Since the normal functions used in electrical engineering work are functions in the set B, most normal functions of interest to engineers are embedded in the set G of generalized functions.

The application of the generalized mathematical system developed in this work is considered in Chapter IV.

The consistency of ordinary operations on functions of B with the corresponding generalized operations on members of G has been established as mentioned before. It is of interest to know how the generalized system can be used in the event an ordinary operation on a function b in B is undefined. Let f denote any one of the ordinary operations of differentiation, integration, addition, or multiplication by members of C^{∞}. If b is in B and is also embedded in G, a unique solution for the ordinary operation f(b) is always defined in the following sense. If b is embedded at the generalized function g_t in G and if f(b) is desired, the corresponding generalized operation $f(g_t)$ in G always exists as a unique function in G. Moreover, if the ordinary operation f(b) is defined, the ordinary function f(b) will be embedded in the generalized function $f(g_t)$.

The result of applying any of the four generalized operations defined for G to a member g_t of G is easily found. In each case, the generalized operation on a member of G is another unique generalized function in G. The result of any of the four generalized operations on g_t is the unique generalized function of G containing the sequence of normal functions $\{f(b_n)\}$, where f is the ordinary operation corresponding to the generalized operation in G considered, and where $\{b_n\}$ is any sequence of normal functions in g_t. Thus the mathematical system developed has potential intuitive content in that each operation on a generalized function g_t in G is intimately related to the corresponding ordinary operation on the normal functions of any sequence that characterizes the function g_t.

The solutions of linear constant coefficient differential equations in G (l.c.c.d.e) are considered in Chapter V. A solution of the non-homogeneous l.c.c.d.e. in $G, L_m(x) = g_t$, satisfying the boundary conditions $\xi_1, \ldots, \xi_m$ at a_1 is defined to be any member function of G, say g_t, that contains the sequence $\{\psi_m(t)\}$ defined as follows. For any $\{b_n\}$ in g_t and for each $n = 1, 2, \ldots, \psi_n(t)$ is the unique solution of the ordinary non-homogeneous l.c.c.d.e. $L_m(x) = b_n(t)$ where $\psi_m^i(a_1) = \xi_{i+1}$ for each $i = 0, \ldots, m-1$.

For two equivalent sequences $\{b_n\}$ and $\{b_n^*\}$ in S_{a_1}, it is found that the corresponding sequences $\{\psi_n\}$ and $\{\psi_n^*\}$ are equivalent in S_{a_1}, where for each $n = 1, 2, \ldots, \psi_n$ and ψ_n^* are respectively, the unique solutions of $L_m(x) = b_n$ and $L_m(x) = b_n^*$ that satisfy the boundary conditions $\psi_n^i(a_1) = \xi_{i+1}$ and $\psi_n^{i*}(a_1) = \xi_{i+1}$ for each $i = 0, \ldots, m-1$. Then the generalized solution $g_{t_{n.h.}}$ of $L_m(x) = g_t$ determined by a sequence $\{b_n\}$ in g_t is independent of the particular sequence $\{b_n\}$ of g_t selected. Therefore, the solution $g_{t_{n.h.}}$ in G corresponding to a given g_t in G is the <u>unique</u> solution in G of the equation $L_m(x) = g_t$ satisfying the boundary conditions $\xi_1, \ldots, \xi_m$ at a_1.

The generalized solution in G for a non-homogeneous system of m l.c.c.d.e. excited by m functions in G is also considered in Chapter V. Using the development for the solution of the single equation $L_m(x) = g_t$, the method of solution for a determinate non-homogeneous system of equations is obtained. For a given set $g_{t1}, \ldots, g_{tm} \in G$ of excitations for the m^{th} order determinate system, a unique solution $g_1, \ldots, g_m$ is shown to exist in G for a sufficient set of boundary conditions specified at the point a_1.

The solutions in G of $L_m(x) = g_t$ are considered in Chapter VI for several important excitations g_t of G. For each g_t considered, the solution $g_{t_{n.h.}}$ of $L_m(x) = g_t$ is investigated to determine if a normal function of B is embedded in the solution.

If g_t is the impulse function, the response $g_{t_{n.h.}}$ will contain an embedded function of B. If also g_t is any generalized function at which a normal function of B is embedded, $g_{tn \cdot h}$ will contain an embedded normal function from B. However, if g_t is a generalized derivative of the delta function, the response of $L_m(x) = g_t$ may not have an embedded normal function.

Several examples of the application of the mathematical system developed to electrical systems are given in Chapter VII. A simple system involving differentiation, addition and multiplication of generalized functions by members of C^{∞} is discussed. Also, considered is a single mesh R.L.C. network as well as a double mesh circuit. In each case considered, the results of the analysis using the mathematical system developed here are found consistent with the classical solutions of these problems.

The longer arguments for justifying generalized multiplication in G are given in Appendix I. The arguments for generalized differentiation, integration, and addition are sufficiently brief to be included in Chapter II.

The major mathematical theorems used in developing the mathematical system given here are reproduced in Appendix II.

Proof of membership in G for the sequences discussed in Chapter III is given in Appendix III. The details of the proof are sufficiently long not to be included in Chapter III.

A discussion of the ordinary linear constant coefficient differential equation and some of the analysis related to the solution of the generalized differential equation $L_m(x) = g_t$ are included in Appendix IV.

In the last Appendix, Appendix V, justification for the results obtained in Chapter VI is given.

Review of Distribution Theory

The theory of distributions will be discussed in some detail at this point, since some of the concepts of distribution theory extend to the mathematical system developed in the present work, and some of the results of distribution theory parallel the results of the present research.

The theory of distributions, and other theories of comparable power, have required a radical departure from some of the basic ideas of classical mathematics. The theory of distributions is an outgrowth of the mathematical field of functional analysis. In one dimension, the distribution theory is constructed as follows.

Suppose $\mathfrak{D}$ is the set of infinitely differentiable real valued functions φ vanishing outside some finite interval of the real line. In general, the interval on which φ is non-zero may vary from element to element in $\mathfrak{D}$. For every normal locally integrable function f[9] defined on the real line, the functional $f\langle\varphi\rangle$ on $\mathfrak{D}$ of f is defined by

$$f\langle\varphi\rangle = \int_{-\infty}^{\infty} f(t)\,\varphi(t)dt, \quad \text{with} \quad \varphi \in \mathfrak{D}\,.$$

The functional $f\langle\varphi\rangle$ is an evaluation of f on all elements of $\mathfrak{D}$. In classical analysis, the normal function f is characterized by its values $f(t)$ for all real t. Alternatively, f can be characterized by its evaluations $f\langle\varphi\rangle$ on the elements of $\mathfrak{D}$ to within a null function.(17) That is, if f and g are two normal locally integrable functions possessing the same evaluations on all elements of $\mathfrak{D}$ then f and g can differ only by a null function. If N is a null function, then for any pair of numbers a and b, $a < b$ it is true that

$$\int_a^b |N(t)|dt = 0 .$$

That is, N vanishes almost everywhere.

For the most part, it can be said that the distinction between two functions differing by a null function is unimportant in applied mathematics. It is this fact that permits the use of Laplace transforms in electrical systems. For situations where the distinction is unimportant, the functional $f\langle\varphi\rangle$ is as satisfactory as $f(t)$ for a characterization of the normal function f.

The functional $f\langle\varphi\rangle$ is linear in the sense that for any two numbers C_1 and C_2 and any two functions φ_1 and φ_2 in $\mathfrak{D}$,

$$f\langle c_1\varphi_1 + c_2\varphi_2\rangle = c_1 f\langle\varphi_1\rangle + c_2 f\langle\varphi_2\rangle$$

$f\langle\varphi\rangle$ is also continuous in the sense that $\varphi_n \rightarrow \varphi$ in $\mathfrak{D}$ implies $f\langle\varphi_n\rangle \rightarrow f\langle\varphi\rangle$, in the sense of convergence of numbers, as $n \rightarrow \infty$. Convergence $\varphi_n \rightarrow \varphi$ in $\mathfrak{D}$ is in the following sense. A sequence of functions $\{\varphi_n\}$ in $\mathfrak{D}$ is said to converge to zero if there is a finite interval I such that all φ_n vanish outside of I, and if, for each fixed nonnegative integer k, $\varphi_n^{(k)}(t) \rightarrow 0$ uniformly for all real t as $n \rightarrow \infty$. A sequence of functions $\{\varphi_n\}$ converges to φ in $\mathfrak{D}$ if the sequence $\{\varphi_n - \varphi\}$ converges to zero in $\mathfrak{D}$.

Every locally integrable normal function f determines uniquely a linear continuous functional on the set of test functions $\mathfrak{D}$, and conversely such a normal function is determined up to a null function by the linear continuous functional it generates.

There are many linear continuous functionals on $\mathfrak{D}$ that are not generated by locally integrable normal functions. For example, if k is any non-negative integer, the equations

$$\delta \langle \varphi \rangle = \varphi(0) \quad \text{and} \quad \delta^{(k)} \langle \varphi \rangle = (-1)^k \varphi^{(k)}(0)$$

assign numbers to each φ in $\mathfrak{D}$ and thereby determine functionals on $\mathfrak{D}$ that are both linear and continuous. However, for any given $k = 0, 1, 2, \ldots$ there does not exist a locally integrable function that generates the functional $\delta^{(k)} \langle \varphi \rangle$, $\varphi \in \mathfrak{D}$.

Linear continuous functionals defined on $\mathfrak{D}$ are called distributions and the set of distributions is the set of generalized functions considered in distribution theory. In distribution theory the linear continuous functional $\delta^{(k)} \langle \varphi \rangle$, where $\varphi \in \mathfrak{D}$, is defined to be the k^{th} derivative of the impulse or δ-function.

The set $\mathfrak{D}^*$ of all distributions forms a vector space, and this space contains the generalized functions corresponding to all locally integrable normal functions as well as the impulse function and all orders of its derivatives.

The operations of differentiation, integration, addition and a form of multiplication can be defined in $\mathfrak{D}^*$. These "generalized" operations are found to be consistent with the ordinary operations of differentiation, integration, addition and multiplication when these ordinary operations on normal functions are defined. It is important to note in particular that every distribution in $\mathfrak{D}^*$ possesses all orders of distribution derivatives in $\mathfrak{D}^*$.

Two distributions T and S in $\mathfrak{D}^*$ are equal if

$$S\langle\varphi\rangle = T\langle\varphi\rangle \quad \text{for every} \quad \varphi \in \mathfrak{D}$$

Multiplication of a distribution T by a real number C, and the addition of S and T are defined by

$$(cT)\langle\varphi\rangle = cT\langle\varphi\rangle, \quad \varphi \in \mathfrak{D}$$

and $$(S + T)\langle\varphi\rangle = S\langle\varphi\rangle + T\langle\varphi\rangle, \quad \varphi \in \mathfrak{D}, \quad \text{respectively.}$$

In general, the product of two distributions in $\mathfrak{D}^*$ is not defined. However, if $\theta \in \mathfrak{D}^*$ is a distribution generated by an infinitely differentiable real valued function, and if T is <u>any</u> member of $\mathfrak{D}^*$, the distribution product $\theta \cdot T$ is defined by

$$(\theta \cdot T)\langle\varphi\rangle = T\langle\theta(t) \cdot \varphi(t)\rangle$$

where $\theta(t)$ is the function generating $\theta\langle\varphi\rangle$ and $\varphi \in \mathfrak{D}$.

If f is a differentiable function and if f' is locally integrable, the normal functions f and f' generate the distributions $f\langle\varphi\rangle$ and $f'\langle\varphi\rangle$ respectively. But

$$f' \langle \varphi \rangle = \int_{-\infty}^{\infty} f'(t)\ \varphi(t)dt$$

$$= - \int_{-\infty}^{\infty} f(t)\ \varphi'(t)dt = -f \langle \varphi' \rangle$$

Then a definite relation exists between $f' \langle \varphi \rangle$ and $f \langle \varphi \rangle$. The distribution $f' \langle \varphi \rangle$ is defined as the derivative of the distribution $f \langle \varphi \rangle$.

If T is an arbitrary distribution in $\mathfrak{D}^*$, the derivative T' is defined by

$$T' \langle \varphi \rangle = T \langle -\varphi' \rangle \ , \quad \varphi \in \mathfrak{D}$$

For derivatives of higher order this definition implies

$$T^k \langle \varphi \rangle = T \langle (-1)^k \varphi^k \rangle , \ \varphi \in \mathfrak{D}$$

It is seen from this definition that every distribution in $\mathfrak{D}^*$ possesses all orders of distribution (generalized) derivatives in $\mathfrak{D}^*$.

Antiderivatives of distributions, corresponding to indefinite integrals of normal functions, can also be defined. The distribution T_1 is called the antiderivative of T if $T_1' \langle \varphi \rangle = T \langle \varphi \rangle$. It is possible to show that every distribution in $\mathfrak{D}^*$ possesses at least one antiderivative in $\mathfrak{D}^*$.

Corresponding to the situation found for indefinite integrals of normal functions, it is found that a distribution may possess more than one antiderivative. A constant c generates the distribution

$$c \langle \varphi \rangle = c \int_{-\infty}^{\infty} \varphi(t)dt, \quad \varphi \in \mathfrak{D} \ .$$

Such a distribution $c\langle\varphi\rangle$ is called a constant distribution. The condition

$$T'\langle\varphi\rangle = 0$$

implies that $T\langle\varphi\rangle$ is a constant distribution. If $T_1\langle\varphi\rangle$ is an antiderivative of $T\langle\varphi\rangle$ and if $c\langle\varphi\rangle$ is any constant distribution it follows that

$$(T_1 + c)\langle\varphi\rangle$$

is also an antiderivative of $T\langle\varphi\rangle$. Any two antiderivatives of a given $T\langle\varphi\rangle$ in $\mathfrak{D}^*$ may differ only by a constant distribution $c\langle\varphi\rangle$.

A sequence of distributions $\{T_n\}$ converges provided that, for every φ in $\mathfrak{D}$ the sequence of numbers $T_n\langle\varphi\rangle$ converges. If $\{T_n\}$ is a convergent sequence of distributions, then for each φ, $\lim_{n\to\infty} T_n\langle\varphi\rangle$ exists and defines a functional $T\langle\varphi\rangle$ on $\mathfrak{D}$. The functional T is linear and it can also be shown that T is continuous. Then a convergent sequence in $\mathfrak{D}^*$ has a limit in $\mathfrak{D}^*$.

Since every locally integrable normal function determines a distribution, convergence of distributions may be used to define generalized limits of sequences of normal functions. In this definition, the generalized limit of a sequence of normal functions, if it exists, is in general a distribution rather than a normal function. The connection between ordinary pointwise limits and generalized limits of sequences of normal functions is not simple. Either of these limits may exist when the other does not, and even when both limits exists, they may not be equal.

If T is an arbitrary distribution in $\mathfrak{D}^*$, it can be shown that at least one sequence $\{f_n(t)\}$ of infinitely differentiable normal functions exists such that the sequence of distributions $\{f_n\langle\varphi\rangle\}$ generated by $\{f_n(t)\}$ converges to T in the sense of convergence in $\mathfrak{D}^*$.[18] It can be seen then that a sequence of infinitely differentiable normal functions $\{\delta_n(t)\}$ exists that has the generalized limit $\delta\langle\varphi\rangle$. No locally integrable normal function exists that can generate the distribution $\delta\langle\varphi\rangle$, however. If $\{\delta_n(t)\}$ converges to the generalized limit $\delta\langle\varphi\rangle$, $\varphi \varepsilon \mathfrak{D}$, it can be shown that $\{\delta_n^k(t)\}$ has the generalized limit $\delta^k\langle\varphi\rangle$, $\varphi \varepsilon \mathfrak{D}$.[19,20,21,22,23]

The concept of a distribution as the solution of a differential equation presents an immediate generalization of the theory of differential equations.[24,25,26] As an example the equation

$$y^{(n)}(t) + \ldots + a_n(t)\ y(t) = f(t)$$

will be considered where a_i, $i = 1,\ldots,n$, are infinitely differentiable functions of t, and f is a locally integrable normal function.

In light of the relation

$$f\langle\varphi\rangle = \int_{-\infty}^{\infty} f(t)\ \varphi(t)\ dt$$

the ordinary differential equation $L_n(y) = f$ can be considered as an equation in distributions.

If T is an arbitrary distribution in $\mathfrak{D}^*$ and if $A_1, A_2, \ldots, A_n$ are the distributions generated by $a_1,\ldots,a_n$ respectively, the solution of

$$Y^{(n)}\langle\varphi\rangle + A_1\langle\varphi\rangle\ Y^{(n-1)}\langle\varphi\rangle + \ldots + A_n\langle\varphi\rangle\ Y\langle\varphi\rangle = T\langle\varphi\rangle$$

where $\varphi \in \mathfrak{D}$, is the sum of a particular distribution solution and the classical solution of the homogeneous equation $L_n(y) = 0$.

CHAPTER II

THE GENERALIZED MATHEMATICAL SYSTEM

Development of the System

In the following work, a generalized mathematical system is developed which contains the improper functions $\delta^{(k)}(t)$, $k = 0,1,2,\ldots,$ and in which are embedded many of the normal functions used in electrical engineering problems. In the mathematical system obtained, there are developed consistent generalized operations of addition, differentiation, integration, and multiplication by normal functions of the set C^{∞}. Again the set of normal functions C^{∞} are those functions that have all orders of derivatives everywhere on the real line E_1.

As in the theory of distributions it appears that a method for generalizing the product of any two members of the set of generalized functions G is not at all apparent. The generalized multiplication that will be developed here is the product of any member function of G by any member function of C^{∞}. It is to be noted, however, that this more restricted concept of generalized multiplication for the mathematical system to be developed is as extensive a concept as the generalized multiplication proposed in the distribution theory of Schwartz. In the distribution theory, the product of two distributions can be defined in a meaningful way as long as one distribution corresponds to some infinitely differentiable function in C^{∞}.

The difficulties involved in developing the generalized mathematical system discussed here lie in the simultaneous selection of the following three items.

(i) A set B of normal functions that are to be extended to a set of generalized functions G.

(ii) A subset S of the set $\mathcal{B}$ of all infinite sequences of the form $\{b_n\}$, $n = 1, 2, \ldots$, that can be composed from the set B.

(iii) An equivalence relation that will partition S in such a way that, at least, the improper functions $\delta^{(k)}$, $k = 0, 1, 2, \ldots$, are members of the set G of generalized functions obtained by partitioning S into disjoint subsets of equivalent sequences and such that many normal functions used in electrical engineering are embedded in G.

In some respects the development of the mathematical system presented in this work parallels the following representation of real numbers by infinite sequences of rational numbers. For a discussion of rational and irrational number systems the reader is referred to Principles of Mathematical Analysis by W. Rudin[27]. The irrational number π could be represented by the sequence of rational numbers

$$\{3, 3.1, 3.14, 3.141, 3.1416, \ldots\}.$$

Another representation of π might be the sequence

$$\{4, 3.1, 3.14, 3.141, 3.1416, \ldots\}.$$

The above two representations would have to have the convergence property

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \pi$$

where a_n and b_n are the $n^{\underline{th}}$ rational numbers in the first and second sequences, respectively.

If R is the set of all rational numbers and if $\mathcal{R}$ is the set of all countably infinite sequences of rationals formed from R, the irrational number π may be represented by a family of sequences of rational numbers. The limit operation

$$\lim_{n \to \infty} a_n$$

considered for all $\{a_n\}$ in $\mathcal{R}$ will partition $\mathcal{R}$ into equivalent disjoint subsets (families) of sequences of rational numbers. The unique family of sequences that contains the sequence

$$\{3,\ 3.1,\ 3.14,\ 3.141,\ 3.1416, \ldots\}$$

that converges to π would be the family of sequences representing π.

In this type of rational-to-irrational extension, the set R is analogous to set B described in (i). The set $\mathcal{R}$ is analogous to set $\mathcal{B}$ of (ii). Finally the limit operation that partitions $\mathcal{R}$ into disjoint families of sequences of rationals is analogous to the equivalence relation described above in statement (iii).

It is found that the selection of S and of the equivalence relation used to partition S are influenced jointly by the types of operations to be generalized from B to G and by the fact that these generalized operations should be meaningful. The generalized operations are to be meaningful in that any disjoint subset of S determined by the selected equivalence relation for S should have the following property. If any disjoint subset of S, determined by the selected equivalence relation for S, is operated on by any one of the generalized operations defined for S, the result should be some unique disjoint subset of S. It is desired that an operation on any disjoint subset of equivalent sequences in S result in a disjoint subset of equivalent sequences in S for the following reason. The purpose of developing a generalized mathematical system of the type considered here is to obtain a system in which operations on any function in the system are <u>always</u> defined in the system. The result of a generalized operation on any disjoint subset of equivalent sequences of S should result in a unique disjoint subset of S for two reasons. First, the generalized operations on the generalized functions should be consistent with corresponding ordinary operations on normal functions when the latter exist. Second, if an electrical system is excited by a generalized function such as an impulse, the response of the system will be single valued, that is there is only one response for the given system corresponding to the impulse excitation.

The factors influencing the determination of the generalized mathematical system and the interrelationship of these factors are shown in Figure 2. It is seen that the generalization procedure depends upon a consistent selection of sets B and S, and of a <u>useful</u> equivalence relation used to partition S into disjoint families of sequences called the generalized functions of the system. The selection of sets B, S and the equivalence relation for S are made when a group of operations are given to be extended to the set of generalized functions G; where certain generalized functions are required to be in G and where certain normal functions are to be embedded in set G.

In the mathematical system developed in this work a normal function b is defined to be in the set of normal functions B if b is a real valued function defined on the real line E_1 such that all orders of

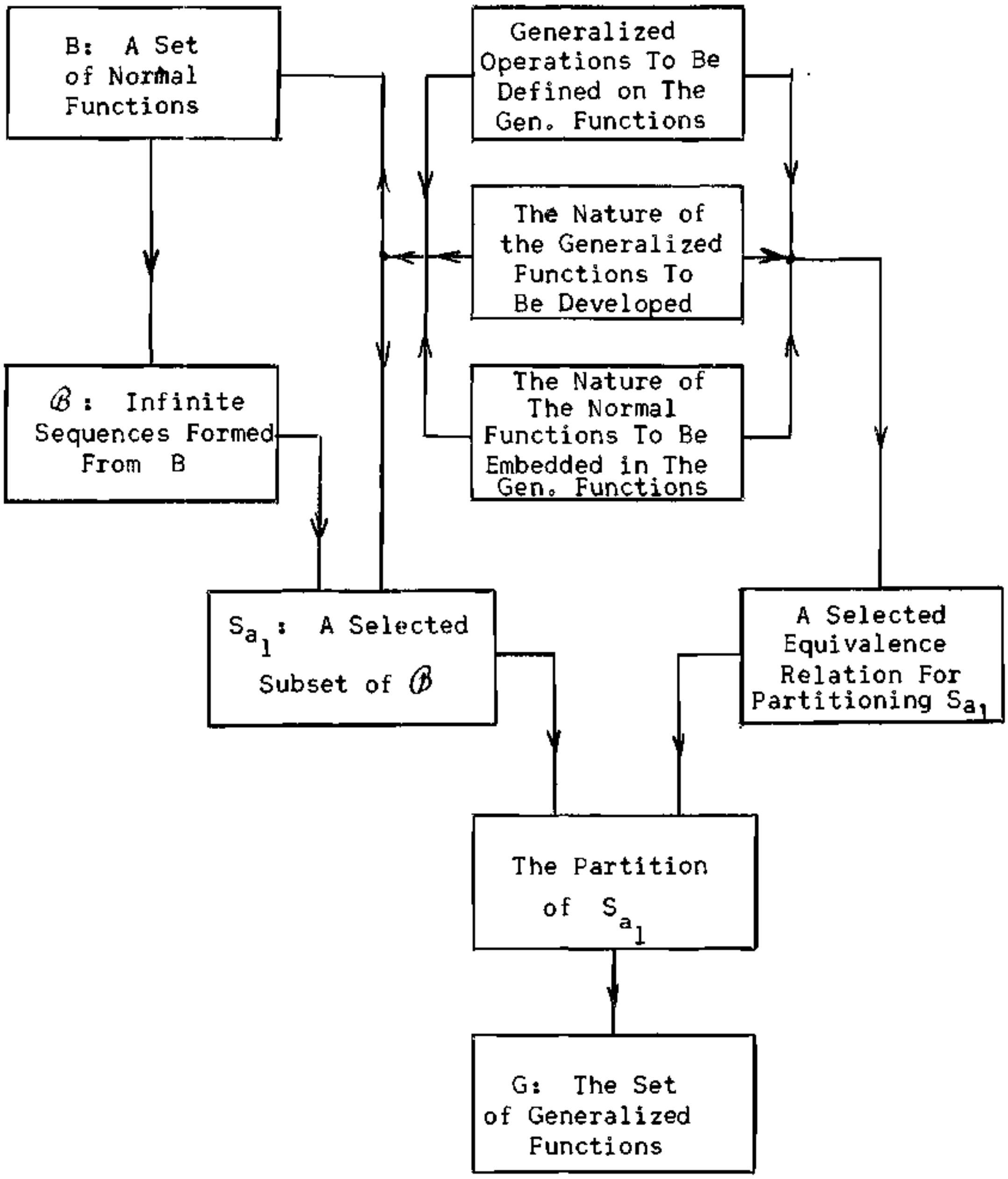

Figure 2. The Interrelationship of the Factors Influencing the Determination of The Mathematical System.

derivatives of b exist except possibly at a finite number of isolated points in any finite interval $[a, b]$. Recalling the additive property of Riemann integrals and recalling that the Riemann integral of any continuous function exists, it must follow that any member of B is Riemann integrable on every finite interval $[a, b]$ in E_1.

For any function b in B and for an arbitrarily selected real constant a_1 in E_1, the operation f^k is defined on B as follows. For any integer k,

$$f^k(b) = \begin{cases} \int_{a_1}^{x_1} \cdots \int_{a_1}^{x_k} b(x_{k+1})\, dx_{k+1} \cdots dx_2, & \text{for } k \geq 1 \\ b(x_1) & \text{, for } k = 0 \\ D_{x_1}^{(-k)}(b) \text{ when the derivative is defined and is zero otherwise} & \text{, for } k \leq -1 \end{cases} \quad (2\text{-}1)$$

On any finite interval $[a, b]$ in E_1, a function b in B is made up of a finite number of pieces of functions belonging to C^∞. Then by the nature of the operation f^k, the set B must be closed with respect to f^k. That is, if k is any integer and if $b \in B$, then $f^k(b) \in B$.

For any given point $x_1 \in E_1$ and for any integer k, the value of $f^k(b)$ at the point x_1 is denoted by $f^k_{x_1}(b)$.

The set $\mathcal{B}$ is defined to be the set of all infinite sequences of the form $\{b_n\}$ where b_n is in B for each $n = 1, 2, \ldots$. If $\{b_n\} \in \mathcal{B}$ then $\{f^k(b_n)\} \in \mathcal{B}$ for any integer k. This follows since B is closed with respect to f^k.

The set S_{a_1} is defined to be the subset of $\mathcal{B}$ such that a sequence $\{b_n\}$ is in S_{a_1} if

(1) $b_n \varepsilon C^{\infty}$ for each $n = 1, 2, \ldots$ (2-2)

(2) For each point x_1 in each interval $[a_1, x_o] \varepsilon E_1$, excluding possibly a finite number of points different from a_1, the limit $\lim_{n \to \infty} f^k_{x_1}(b_n)$ exists in E_1 for each integer k. The finite number of exceptional points in $[a_1, x_o]$ is denoted by $C^k_{b_n}(a_1, x]$.

(3) There exists an integer $k_{b_n} \geq 0$ such that the sequence $\left\{f^{k_{b_n}}(b_n)\right\}$ is boundedly convergent (AII) and the limit function $\lim_{n \to \infty} f^{k_{b_n}}(b_n)$ exists and is Riemann integrable on each interval $[a_1, x_o]$ in E_1.

Since in general S_{a_1} will depend on the constant a_1 selected in (2-1), S_{a_1} carries the subscript a_1.

The symbol $[a_1, x_o]$ will be used to denote the closed interval between a_1 and x_o when $a_1 < x_o$ or when $a_1 > x_o$.

Two sequences $\{b_n\}$ and $\{b^*_n\}$ in S_{a_1} are defined to be equivalent and are written $\{b_n\} \sim \{b^*_n\}$ in S_{a_1} if for each integer k,

$$\lim_{n \to \infty} \left| f^k_{x_1}(b_n) - f^k_{x_1}(b^*_n) \right| = 0, \qquad (2\text{-}3)$$

for all points x_1 in E_1, except possibly a finite number of points different from a_1 in each $[a_1, x_o] \varepsilon E_1$. The set of possible points in $[a_1, x_o]$ where the limit (2-3) differs from zero will be denoted by $d^k_{b_n, b^*_n}(a_1, x_o]$.

Equation (2-3) partitions S_{a_1} into disjoint subsets of equivalent sequences. These families of equivalent sequences are described by the set $G = \{g_t | t \varepsilon T\}$ where for each t in the index T, g_t is one of the disjoint families of sequences.

The sets g_t, $t \varepsilon T$, are defined to be the generalized functions for the mathematical system developed here.

With the equivalence relation (2-3), each generalized function g_t in G can be extended to a set of equivalent sequences $\hat{g}_t$ in $\mathcal{B}$. That is, a sequence $\{f_n\}$ in the set $\mathcal{B}$ will be in the extension $\hat{g}_t$ of the generalized function g_t if $\{f_n\}$ is equivalent to some sequence $\{b_n\}$ in g_t. If a sequence $\{f_n\}$ in $\mathcal{B}$ is equivalent under (2-3) to any sequence $\{b_n\}$ in g_t then it is equivalent to every sequence in g_t. Also, if $\{f_n\}$ in set $\mathcal{B}$ is equivalent to the sequence $\{b_n\}$ in g_t, it can not be equivalent to any sequence in any other generalized function of G except g_t. This follows since the equivalence of $\{f_n\}$ to $\{b_n\}$ and $\{f_n\}$ to $\{b_n^*\}$ where $\{b_n\}$ is in g_{t_1} and $\{b_n^*\}$ is in g_{t_2} would imply that $\{b_n\}$ and $\{b_n^*\}$ are equivalent sequences of S_{a_1} and hence that $g_{t_1} = g_{t_2}$. Then the extensions $\hat{g}_t$, where $t \varepsilon T$, of the generalized functions g_t are also disjoint families of sequences in $\mathcal{B}$.

Figure 2, indicates the partition of S_{a_1} from which the generalized functions G are obtained. The extensions in $\mathcal{B}$ of the generalized functions g_t where $t \varepsilon T$, are also indicated in the figure.

For any g_t in G the extension $\hat{g}_t$ of g_t is unique. Any arbitrary sequence $\{f_n\}$ in the unique $\hat{g}_t$ corresponding to g_t is defined to be a representation for the generalized function g_t.

A normal function b in B is defined to be embedded in G if

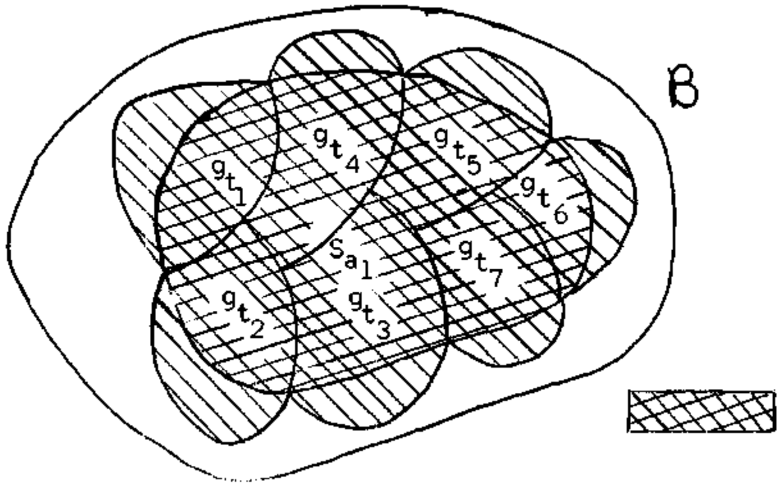

Figure 3. Extensions of g_t To $\hat{g}_t$ In $\mathcal{B}$.

the constant sequence $\{b_n\}$, where each $b_n = b$, is in some $\hat{g}_t$ where $t \varepsilon T$. That is, b is embedded in G if the constant sequence $\{b_n\}$, where each $b_n = b$, belongs to the union $\bigcup_{t \varepsilon T} \hat{g}_t$. If b is embedded in G, the sequence $\{b_n\}$, where each $b_n = b$, is contained in a unique $\hat{g}_t$. Therefore, if a normal function b in B is embedded in G, it is embedded at some unique member g_t of G.

If $b \varepsilon B$ is such that it is infinitely differentiable at the point a_1, it is always possible to find a sequence of functions in S_{a_1} that is equivalent to the constant sequence $\{b_n\}$, where each $b_n = b$, (A-III, case 5). Then for a given constant a_1 used in the definition of f^k, all members of B that are infinitely differentiable at a_1 are embedded in G.

Generalized Operations In G

Summary of Operations

It will be shown that the following generalized operation may be defined in the set G.

The generalized derivative can be defined in the following way. If g_t is any member of G and if $\{b_n\} \varepsilon g_t$, then the derivative of g_t, denoted by $D^{(1)}(g_t)$ can be defined uniquely as that member of G which is equivalent to $\{D_x^{(1)} b_n\}$. The selection of a particular sequence of g_t in defining $D^{(1)}(g_t)$ is unimportant since the equivalence of two sequences of S_{a_1} implies the equivalence of their corresponding derivative sequences in S_{a_1}.

The generalized integral can be defined as follows. If $g_t \varepsilon G$ and if $\{b_n\} \varepsilon g_t$, the generalized integral of g_t, denoted by the symbol $\int_{a_1}^{x} (g_t)$, can be defined uniquely as that member of G which is equivalent to the sequence $\left\{\int_{a_1}^{x} b_n \, dt\right\}$. The selection of a particular sequence in g_t to consider is unimportant since the equivalence of any two sequences in S_{a_1} implies the equivalence of their corresponding integral sequences in S_{a_1}.

Generalized addition can be defined as follows. If g_{t_1} and g_{t_2} are any two sequences in G with $\{b_n\} \varepsilon g_{t_1}$ and $\{b_n^*\} \varepsilon g_{t_2}$, the generalized sum of g_{t_1} and g_{t_2}, denoted by the symbol $g_{t_1} + g_{t_2}$, can be defined uniquely to be that element of G equivalent to $\{b_n + b_n^*\}$.

The selection of particular $\{b_n\} \varepsilon g_{t_1}$ and of $\{b_n^*\} \varepsilon g_{t_2}$ is unimportant for the definition of addition.

Generalized multiplication is defined in the following way. If $g_t \varepsilon G$ and $\{b_n\} \varepsilon g_t$, the product of g_t by any given member of C^∞ can be defined uniquely to be that member of G, denoted by $g \cdot g_t$, which is equivalent to the sequence $\{gb_n\}$.

The selection of a particular sequence $\{b_n\}$ in g_t in defining $g \cdot g_t$ is unimportant since

$$\{b_n\} \sim \{b_n^*\}$$

in S_{a_1} implies

$$\{gb_n\} \sim \{gb_n^*\}$$

in S_{a_1} whenever $g \varepsilon C^\infty$.

<u>Differentiation</u>

If g_t is any member of G and if $\{b_n\}$ is any sequence of g_t, it can be shown that the sequence $\{D_x' b_n\}$ is in some generalized function of G. Moreover, it can be shown that the member of G that contains $\{D_x' b_n\}$ is the same for any sequence $\{b_n\}$ in g_t.

The generalized derivative of g_t in G is defined to be the unique member of G containing the sequence $\{D_x' b_n\}$, where $\{b_n\}$ is an arbitrary sequence in g_t. The generalized derivative of g_t in G will be denoted by $D^1(g_t)$.

To show that the generalized derivative of any g_t in G exists and is unique in G it is sufficient to show the equivalence

$$\left\{D_x' b_n\right\} \sim \left\{D_x' b_n^*\right\} \text{ in } S_{a_1}$$

given the equivalence

$$\left\{b_n\right\} \sim \left\{b_n^*\right\} \text{ in } S_{a_1}$$

To this end, let

$$\left\{b_n\right\} \varepsilon\ S_{a_1}$$

and consider the sequence

$$\left\{f^k(D_x' b_n)\right\}$$

When $k \le 0$, $f^k(D_x' b_n) = f^{k-1}(b_n)$ for each $n = 1, 2, 3, \ldots$. When $k > 0$, $f^k(D_x' b_n) = f^{k-1}(b_n) - b_n(a_1) \cdot f^{k-1}(1)$ (2-4)

which follow since

$$\int_{a_1}^{x_k} (D'_{x_{k+1}}(b_n))dx_{k+1} = b_n(x_k) - b_n(a_1)$$

The membership of $\left\{b_n\right\}$ in S_{a_1} implies that $\lim_{n \to \infty} b_n(a_1) \varepsilon E_1$. Then $\lim_{n \to \infty} f_{x_1}^k(D_x' b_n) \varepsilon E_1$ for every $x_1 \varepsilon E_1$ except points in the set $c_{b_n}^{k-1}(a_1, x_o]$ in each $[a_1, x_o]$ of E_1 for each integer K.

For $\left\{f^{k_{b_n}}(b_n)\right\}$, where $k_{b_n} \ge 0$ as defined in equation (2-2),

$$f^{k_{b_n}+1}(D_x' b_n) = f^{k_{b_n}}(b_n) - b_n(a_1) \cdot f^{k_{b_n}}(1) \qquad (2\text{-}5)$$

for every $n = 1, 2, 3, \ldots$.

For $k > 0$, it is to be noted that

$$f^k(1) = \int_{a_1}^{x_1} \cdots \int_{a_1}^{x_k} (1)\, dx_{k+1} \cdots dx_2$$

Since $\lim_{n \to \infty} b_n(a_1)$ exists in E_1, some upper bound exists for the collection $|b_n(a_1)|$, $n = 1, 2, 3, \ldots$. Then from condition (3) of equation (2-2) for membership of $\{b_n\}$ in S_{a_1}, it follows that an integer $K_{D' b_n} = K_{b_n} + 1 \geq 0$ does exist such that condition (3) of (2-2) holds for the sequence $\left\{ f^{K_{D' b_n}}(D'_x\, b_n) \right\}$. Then if $\{b_n\} \varepsilon S_{a_1}$, it must follow that $\{D'_x\, b_n\} \varepsilon S_{a_1}$.

If $\{b_n\} \sim \{b_n^*\}$ in S_{a_1} and if $k \leq 0$,

$$\lim_{n \to \infty} |f^k_{x_1}(D'_x b_n) - f^k_{x_1}(D'_x\, b^*_n)| = \lim_{n \to \infty} |f^{k-1}_{x_1}(b_n) - f^{k-1}_{x_1}(b^*_n)| = 0 \quad (2\text{-}6)$$

for all points x_1 not in $d^{k-1}_{b_n b^*_n}\,(a_1, x_o]$ for each interval $[a_1, x_o]$ in E_1.

For the integers $k > 0$, there is the inequality

$$0 \leq |f^k_{x_1}(D'_x b_n) - f^k_{x_1}(D'_x\, b^*_n)| \leq$$

$$|f^{k-1}_{x_1}(b_n) - f^{k-1}_{x_1}(b^*_n)| + |f^{k-1}_{x_1}(1)| \cdot |b_n(a_1) - b^*_n(a_1)|$$

for each n and each point x_1.

Hence since $\lim_{n \to \infty} |b_n(a_1) - b^*_n(a_1)| = 0$, it must follow that

$$\lim_{n \to \infty} |f^k_{x_1}(D'_x\, (b_n)) - f^k_{x_1}(D'_x(b^*_n))| = 0 \qquad (2\text{-}7)$$

for all x_1 in E_1 except for x_1 in $d_{b_n b_n^*}^{(k-1)}(a_1, x_o]$ for each given interval $[a_1, x_o]$ in E_1. Therefore

$$\left\{D'_x\ b_n\right\} \sim \left\{D'_x\ b_n^*\right\} \text{ in } S_{a_1} \text{ if } \left\{b_n\right\} \sim \left\{b_n^*\right\} \text{ in } S_{a_1} \quad (2\text{-}8)$$

Integration

If g_t is any member of G and if $\left\{b_n\right\}$ is any sequence of g_t, it can be shown that the sequence $\left\{\int_{a_1}^{x} b_n\, dt\right\}$ is in some generalized function of G. Also, it can be shown that the member of G that contains $\left\{\int_{a_1}^{x} b_n\, dt\right\}$ is the same for any sequence $\left\{b_n\right\}$ in g_t.

The generalized integral of g_t in G is defined to be the unique member of G containing the sequence $\left\{\int_{a_1}^{x} b_n\, dt\right\}$, where $\left\{b_n\right\}$ is an arbitrary sequence in g_t. The generalized integral of g_t in G will be denoted by $\int_{a_1}^{x}(g_t)$.

To show that the generalized integral of any g_t in G exists and is unique in G, it is sufficient to show the equivalence

$$\left\{\int_{a_1}^{x} b_n\, dt\right\} \sim \left\{\int_{a_1}^{x} b_n^*\, dt\right\} \text{ in } S_{a_1}$$

given the equivalence

$$\left\{b_n\right\} \sim \left\{b_n^*\right\} \text{ in } S_{a_1}$$

To show this, let $\left\{b_n\right\} \varepsilon\ S_{a_1}$ and consider the sequence $\left\{\int_{a_2}^{x} b_n\, dt\right\}$ where a_2 is some given constant in E_1, but where a_2 possibly is different from the constant a_1 specified in the equation (2-1).

Since each $b_n \varepsilon C^\infty$, it follows that $\int_{a_2}^{x} b_n \, dt \, \varepsilon \, C^\infty$ also. If $k \geq 0$, there is for each member of

$$\left\{ f^k \left(\int_{a_2}^{x_{k+1}} b_n \, dx_{k+2} \right) \right\}$$

the expansion

$$f^k \left(\int_{a_2}^{x_{k+1}} b_n \, dx_{k+2} \right) = f^{k+1}(b_n) - \left(\int_{a_1}^{a_2} b_n \, dt \right) \cdot f^k(1) \quad (2\text{-}9)$$

which follows since

$$\int_{a_1}^{a_2} b_n \, dt$$

is some constant for each given n. If $k < 0$, then for each n,

$$f^k \left(\int_{a_2}^{x} b_n \, dt \right) = D_{x_1}^{-(k+1)}(b_n) \quad (2\text{-}10)$$

In condition (2) of (2-2) for membership of $\{b_n\}$ in S_{a_1}, it is possible to have $\lim_{n\to\infty} \int_{a_1}^{a_2} b_n \, dt \notin E_1$. Then in general, the condition (2) of (2-2) will hold for $\left\{ \int_{a_2}^{x} b_n \, dt \right\}$ only when $a_1 = a_2$ such that $\int_{a_1}^{a_2} b_n \, dt = 0$ for every $n = 1, 2, \ldots$. From condition (3) of (2-2), it is seen that an integer $k \int_{a_1}^{x} b_n \, dt \geq 0$ does exist such that condition (3) of (2-2) holds for $\left\{ \int_{a_1}^{x} b_n \, dt \right\}$. In general, this is not the case for $\left\{\int_{a_2}^{x} b_n\, dt\right\}$ if $a_2 \neq a_1$, since $\lim_{n\to\infty} \int_{a_1}^{a_2} b_n\, dt$ may diverge.

In general $\{b_n\} \varepsilon\, S_{a_1}$ implies

$$\left\{\int_{a_2}^{x} b_n\, dt\right\} \varepsilon\, S_{a_1} \quad \text{if} \quad a_1 = a_2 \tag{2-11}$$

If $\{b_n\} \sim \{b_n^*\}$ in S_{a_1} and if $k < 0$,

$$\lim_{n\to\infty} \left| f_{x_1}^{k}\left(\int_{a_2}^{x_{k+1}} b_n\, dt\right) - f_{x_1}^{k}\left(\int_{a_2}^{x_{k+1}} b_n^*\, dt\right)\right| =$$

$$\lim_{n\to\infty} |D_{x_1}^{-(k+1)}(b_n) - D_{x_1}^{-(k+1)}(b_n^*)| = 0$$

for all x_1 in E_1 except for x_1 in $d_{b_n b_n^*}^{(k+1)}(a_1, x_o]$ for each interval $[a_1, x_o]$ in E_1.

For $k \geq 0$, there is the following inequality

$$0 \leq \left| f_{x_1}^{k}\left(\int_{a_2}^{x_{k+1}} b_n\, dt\right) - f_{x_1}^{k}\left(\int_{a_2}^{x_{k+1}} b_n^*\, dt\right)\right| \leq$$

$$\left|\int_{a_1}^{a_2} b_n\, dt - \int_{a_1}^{a_2} b_n^*\, dt\right| + \left| f_{x_1}^{k+1}(b_n) - f_{x_1}^{k+1}(b_n^*)\right|$$

for each point x_1 and each positive integer n.

Then if $k \geq 0$ and if $a_2 = a_1$,

$$\lim_{n\to\infty} \left| f_{x_1}^{k}\left(\int_{a_1}^{x_{k+1}} b_n\, dt\right) - f_{x_1}^{k}\left(\int_{a_1}^{x_{k+1}} b_n^*\, dt\right)\right| = 0$$

for all x_1 in E_1 except for x_1 in the set $d^{k+1}_{b_n b^*_n}(a_1, x_o]$ for each given $[a_1, x_o]$ in E_1.

It follows then that

$$\left\{\int_{a_2=a_1}^{x} b_n \, dt\right\} \sim \left\{\int_{a_2=a_1}^{x} b^*_n \, dt\right\} \text{ in } S_{a_1} \qquad (2\text{-}12)$$

whenever $\{b_n\} \sim \{b^*_n\}$ in S_{a_1}. In general this result holds only for $a_2 = a_1$.

Addition

If g_{t_1} and g_{t_2} are any two members of G and if $\{b_n\}$ is any sequence of g_{t_1} and $\{b^*_n\}$ any sequence of g_{t_2}, it can be shown that the sequence $\{b_n + b^*_n\}$ is in some generalized function of G. Moreover, it can be shown that the member of G containing the sequence $\{b_n + b^*_n\}$ is the same for any pair of sequences $\{b_n\}$ and $\{b^*_n\}$ selected respectively from g_{t_1} and g_{t_2}.

The generalized sum of g_{t_1} and g_{t_2} in G is defined to be the unique member of G containing the sequence $\{b_n + b^*_n\}$ where $\{b_n\}$ is any sequence of g_{t_1} and $\{b^*_n\}$ is any sequence of g_{t_2}. The generalized sum of g_{t_1} and g_{t_2} in G will be denoted by $g_{t_1} + g_{t_2}$.

To show that the generalized sum of any g_{t_1} and g_{t_2} in G exists and is unique in G, it is sufficient to show that

$$\{b_n\} \sim \{c_n\} \text{ in } S_{a_1} \text{ and } \{b^*_n\} \sim \{c^*_n\} \text{ in } S_{a_1}$$

imply

$$\{b_n + b^*_n\} \sim \{c_n + c^*_n\} \text{ in } S_{a_1}.$$

Since $\{b_n\}$ and $\{b_n^*\}$ are assumed to be in S_{a_1}, $b_n + b_n^*$ is a member of C^∞.

Since

$$f^k(b_n + b_n^*) = f^k(b_n) + f^k(b_n^*)$$

the properties (2) and (3) of (2-2) hold for $\{b_n + b_n^*\}$. That is $\{b_n + b_n^*\} \varepsilon S_{a_1}$ if $\{b_n\}$ and $\{b_n^*\}$ belong to S_{a_1}.

With $\{b_n\} \sim \{c_n\}$ and $\{b_n^*\} \sim \{c_n^*\}$ in S_{a_1}, the inequality

$$0 \leq |f_{x_1}^k(b_n + b_n^*) - f_{x_1}^k(c_n + c_n^*)| \leq$$

$$|f_{x_1}^k(b_n) - f_{x_1}^k(c_n)| + |f_{x_1}^k(b_n^*) - f_{x_1}^k(c_n^*)|$$

for each x_1 and each k, implies

$$\lim_{n \to \infty} |f_{x_1}^k(b_n + b_n^*) - f_{x_1}^k(c_n + c_n^*)| = 0$$

for all x_1 in E_1 not contained in

$$d_{b_n c_n}^k(a_1, x_o] \quad \text{or} \quad d_{b_n^* c_n^*}^k(a_1, x_o]$$

for each interval $[a_1, x_o]$ in E_1.

Then if $\{b_n\} \sim \{c_n\}$ in S_{a_1} and $\{b_n^*\} \sim \{c_n^*\}$ in S_{a_1},

$$\{b_n + b_n^*\} \sim \{c_n + c_n^*\} \text{ in } S_{a_1} \tag{2-13}$$

<u>Multiplication</u>

If g_t is any member of G and if $\{b_n\}$ is any sequence of g_t, it can be shown that the sequence $\{gb_n\}$ is in some generalized function of G if it is known that g is a normal function in the set C^∞. In addition, for such a normal function g, it can be shown that the member of G containing the sequence $\{gb_n\}$ is the same for any sequence $\{b_n\}$ in the generalized function g_t.

If $g \in C^\infty$, the generalized product of g and g_t in G is defined to be the unique member of G containing the sequence $\{g \circ b_n\}$, where $\{b_n\}$ is an arbitrary sequence in g_t. The generalized product of g and g_t in G will be denoted by $g \circ g_t$, where $g \in C^\infty$.

To show that the generalized product of any g_t in G and any $g \in C^\infty$ exists and is unique in G it is sufficient to show that the equivalence

$$\{b_n\} \sim \{b_n^*\}$$

in S_{a_1} implies the equivalence

$$\{g \circ b_n\} \sim \{g \circ b_n^*\}$$

in S_{a_1} for any given $g \in C^\infty$. To this end $\{b_n\} \in S_{a_1}$ and $g \in C^\infty$ are considered. It will be shown now that $\{gb_n\} \in S_{a_1}$.

Since g as well as each b_n are members of C^∞, the function $g \cdot b_n$ must belong to C^∞ for each n.

For any integer k, $f^k(g \cdot b_n)$ is to be considered.

For $k \leq 1$, $f^k(g \cdot b_n) = D_{x_1}^{(-k)}(g \cdot b_n)$.

For $k = -1$, $D_{x_1}^{(1)}(gb_n) = gb_n^{(1)} + g^{(1)}\, b_n$

$k = -2$, $D_{x_1}^{(2)}(gb_n) = gb_n^{(2)} + 2g^{(1)}\, b_n^{(1)} + g^{(2)}\, b_n$

and for $k = -m$ with m a positive integer,

$$D_x^m(gb_n) = gb_n^{(m)} + mb^{(m-1)}\, g^{(1)} + \dots$$

$$+ \frac{(m)(m-1)\dots(m-j)}{(j+1)!}\, b^{m-(j+1)}\, g^{(j+1)} + \dots g^{(m)} b_n \qquad (2\text{-}14)$$

with $m = 1, 2, \dots,$ and $n = 1, 2, \dots$.

For $k = 0$, $f^o(gb_n) = gb_n$ for each n.

Then for the integers $k \leq = -1$ or $k = 0$, it can be seen that $\lim_{n\to\infty} f_{x_1}^k(gb_n) \varepsilon E_1$ for all x_1 in E_1 except for possibly a finite number of points not equal to a_1 in each $[a_1, x_o] \varepsilon E_1$. That this is the case for $k \leq -1$ can be seen from

$$\lim_{n\to\infty} \frac{m(m-1)\dots(m-j)}{(j+1)!}\, b_n^{m-(j+1)}\, g^{(j+1)} =$$

$$\frac{m(m-1)\dots(m-j)}{(j+1)!}\, g^{(j+1)} \cdot \lim_{n\to\infty} b_n^{m-(j+1)}$$

for each $m = 1, 2, \dots$. Then if $\{b_n\} \varepsilon S_{a_1}$, the second property of (2-2) holds for $\{gb_n\}$ where $g \varepsilon C^\infty$ and $k \leq 0$. Property (2) of (2-2) also holds for each $k > 1$. This fact will now be shown.

For each $k \geq 1$, the function $f^k(gb_n)$ can be expended into a finite number of terms in such a way that the sequence $\{f^k(gb_n)\}$ is the sum of a finite number of sequences (A-I). Each component sequence in the finite sum constructed from $\{f^k(gb_n)\}$ is one of two types. The two possible types of component sequences of the sum are

$$\left\{(-1)^{k+v}\, g^{(-k+v)}\, f^v(b_n)\right\} \qquad (2\text{-}15a)$$

where v is an integer with $v \geq k$; and

$$\left\{(-1)^{q+z} f^{(k-q)} (g^{(z-q)} \cdot f^z(b_n))\right\} \tag{2-15b}$$

where q, z are integers with $0 \leq q \leq k - 1$ and where z can be made arbitrarily large with a sufficiently large expansion of $f^k(gb_n)$ (A-I).

For (2-15a) it can be seen that

$$\lim_{n \to \infty} g^{-k+v} \cdot f^v(b_n) = g^{-k+v} \cdot \lim_{n \to \infty} f^v(b_n)$$

Then

$$\lim_{n \to \infty} g^{-k+v}(x_1) \cdot f^v_{x_1}(b_n) \tag{2-16}$$

exists in E_1 for all $x_1 \varepsilon E_1$ except for possibly a finite number of points $x_1 \neq a_1$ in each $[a_1, x_o]$.

Since z can be made arbitrarily large with a sufficient expansion of $f^k(gb_n)$, let z be made at least as large as k_{b_n} for each term of type (2-15b). Then, as a consequence of the theorems given in Appendix II,

$$\lim_{n \to \infty} f^{k-q}_{x_1} (g^{z-q} \cdot f^z(b_n)) =$$

$$f^{k-q}_{x_1} (g^{z-q} \cdot \lim_{n \to \infty} f^z(b_n)) \tag{2-17}$$

which exists in E_1 for every $x_1 \varepsilon E_1$.

Since the expansion of $\left\{f^k(gb_n)\right\}$ is a finite number of sequences of the types (2-15a) and (2-15b) it must follow that

$$\lim_{n \to \infty} f^k_{x_1}(gb_n)$$

exists in E_1 for all $x_1 \varepsilon E_1$ except possibly a finite number of $x_1 \neq a_1$ in each $[a_1, x_o] \varepsilon E_1$.

With $k = k_{b_n}$ and z taken greater than or equal to k_{b_n} in the terms of types (2-15a) and (2-15b), it is found that

$$\left\{(-1)^{k+v} g^{-k+v} f^v(b_n)\right\} \tag{2-18}$$

is boundedly convergent and the limit of this sequence as $n \to \infty$ is Riemann integrable on each $[a_1, x_o]$, (A-II). Also the sequence

$$\left\{(-1)^{q+z} f^{k-q} (g^{z-q} \cdot f^z(b_n))\right\} \tag{2-19}$$

is boundedly convergent and the limit of this sequence as $n \to \infty$ must be Riemann integrable on each $[a_1, x_o]$ under the conditions that $k = k_{b_n}$ and $z \geq k_{b_n}$, (A-II). Therefore, since the sum of a finite number of boundedly convergent sequences is a boundedly convergent sequence and since the finite sum of Riemann integrable functions is Riemann integrable it must follow that $\left\{f^{k_{b_n}}(gb_n)\right\}$ is boundedly convergent and $\lim_{n \to \infty} f^{k_{b_n}}(gb_n)$ is Riemann integrable on every $[a_1, x_o] \varepsilon E_1$. Then $\left\{f^k(gb_n)\right\}$ satisfies the three conditions of equation (2-2), whenever $\left\{b_n\right\} \varepsilon S_{a_1}$ and $g \varepsilon C^\infty$. Since $b_n g \varepsilon C^\infty$, if $b_n \varepsilon C^\infty$, $\left\{b_n g\right\} \varepsilon S_{a_1}$ if $\left\{b_n\right\} \varepsilon S_{a_1}$ and $g \varepsilon C^\infty$.

Suppose $\left\{b_n\right\} \sim \left\{b^*_n\right\}$ in S_{a_1} and $g \varepsilon C^\infty$. The same expansion of $\left\{f^k(gb_n)\right\}$ holds for $\left\{f^k(gb^*_n)\right\}$. Then with the same expansions for each of these two sequences, each of the terms like (2-18) in the expansion of $\left\{f^k(gb_n)\right\}$ has a corresponding component sequence in the expansion of $\left\{f^k(gb_n^*)\right\}$. Also there will be a one-to-one correspondence between component sequences like (2-19) in the expansion of $\left\{f^k(gb_n)\right\}$ and $\left\{f^k(gb_n^*)\right\}$.

With the expansion of $\left\{f^k(gb_n^*)\right\}$ and of $\left\{f^k(gb_n)\right\}$ extended such that z is greater than or equal to the maximum of k_{b_n} and $k_{b_n^*}$ it is seen that

$$\lim_{n \to \infty} |g^{-k+v}(x_1) \cdot (f_{x_1}^v(b_n) - f_{x_1}^v(b_n^*))| = 0$$

for all x_1 but possibly a finite number of points not equal to a_1 in each $[a_1, x_o]$. Also,

$$\lim_{n \to \infty} \left| f_{x_1}^{k-q} \left(g^{z-q} \cdot (f^z(b_n) - f^z(b_n^*)) \right) \right| = 0$$

for all $x_1 \varepsilon E_1$,(A-I). Then

$$\lim_{n \to \infty} |f_{x_1}^k(gb_n) - f_{x_1}^k(gb_n^*)| = 0$$

for all but possibly a finite number of points different from a_1 in each $[a_1, x_o]$ in E_1.

Therefore, if $\left\{b_n\right\} \sim \left\{b_n^*\right\}$ in S_{a_1} and if $g \varepsilon C^\infty$, the sequences $\left\{gb_n\right\}$ and $\left\{gb_n^*\right\}$ are equivalent in S_{a_1}.

The Zero Function

The generalized function in G that contains the zero sequence $\left\{b_n\right\}$, where each $b_n = 0$, is defined as the zero element of G and it is denoted by g_o. If g_t is any member of G the generalized sum $g_t + g_o$ is the unique member of G containing the sequence $\{b_n + b_n^*\}$, where $\{b_n\}$ is any sequence in g_t and $\{b_n^*\}$ is any sequence in g_o. For the particular sequence $\{b_n^*\} = \{0\}$ in g_o, $\{b_n + b_n^*\} = \{b_n\}$. But $\{b_n\} \varepsilon g_t$. That is, $g_t + g_o = g_t$ for any g_t in G.

The Consistency of Ordinary Operations and Generalized Operations

Differentiation

Consider a normal function $g \varepsilon B$ embedded in G at the member g_t, and suppose $D_x'f$ is defined on E_1. Then $D_x'f$ must be a member of B. For any sequence $\{b_n\}$ in g_t it follows from the embedding definition that $\{b_n\}$ is equivalent to $\{f_n\}$, where $f_n = f$. For any sequence $\{b_n\}$ in g_t the sequence $\{D_x' b_n\}$ is in the generalized function $D'(g_t)$. With reference to equations (2-6) and (2-7) it follows that $\{D_x' b_n\}$ is equivalent to the constant sequence $\{D_x' f_n\}$, $f_n = f$. Then if $f \varepsilon B$ is embedded in G at g_t and if $D_x'f$ is defined on E_1, $D_x'f$ is embedded in G at $D'(g_t)$. In the above sense the operation of generalized differentiation in G is consistent with ordinary differentiation.

Integration

Let $f \varepsilon B$ be a function embedded in G at g_t. The function $\int_{a_1}^{x} f(t)\,dt$ is a member of B if f is in B. For any sequence $\{b_n\}$ in g_t, $\{b_n\}$ is equivalent to the sequence $\{f_n\}$, $f_n = f$. Also if $\{b_n\} \varepsilon g_t$, $\left\{\int_{a_1}^{x} b_n\,dt\right\} \varepsilon \int_{a_1}^{x} (g_t)$. Referring to the section on the derivation of the generalized integral it is found that $\left\{\int_{a_1}^{x} b_n\,dt\right\}$ is equivalent to the constant sequence $\left\{\int_{a_1}^{x} f_n \, dt\right\}$, where $f_n = f$. That is $\int_{a_1}^{x} f \, dt$ is embedded in G at $\int_{a_1}^{x} (g_t)$.

The operation of generalized integration in G is consistent with ordinary integration of ordinary functions in B in the sense that $\int_{a_1}^{x} f \, dt$ is embedded at $\int_{a_1}^{x} (g_t)$ if f is embedded in G at g_t.

<u>Addition</u>

Let f, where $f \varepsilon B$, be embedded at g_{t_1} in G and let h, where $h \varepsilon B$, be embedded at g_{t_2} in G. Then the function $f + h \varepsilon B$. If $\left\{b_n\right\} \varepsilon g_{t_1}$, then $\left\{b_n\right\} \sim \left\{f_n\right\}$, $f_n = f$. If $\left\{b_n^*\right\} \varepsilon g_{t_2}$, then $\left\{b_n^*\right\} \sim \left\{h_n\right\}$, $h_n = h$. The sequence $\left\{b_n + b_n^*\right\}$ must be in $g_{t_1} + g_{t_2}$. Also $\left\{b_n + b_n^*\right\}$ is equivalent to the constant sequence $\left\{f_n + h_n\right\}$ where $f_n = f$ and $h_n = h$. Therefore, $f + h$ is embedded at $g_{t_1} + g_{t_2}$ in G. In this sense, the operation of generalized addition in G is consistent with the ordinary addition of ordinary functions of set B.

<u>Multiplication</u>

Let $f \varepsilon B$ be embedded at g_t in G and let g be any function in C^∞. The function $f \cdot g$ must be contained in set B. For any sequence $\left\{b_n\right\}$ in g_t, $\left\{b_n\right\}$ is equivalent to the sequence $\left\{f_n\right\}$, where $f_n = f$. Moreover, if $g \varepsilon C^\infty$, $\left\{g \cdot b_n\right\}$ belongs to the generalized function $g \cdot g_t$ in G. With reference to the analysis used in the derivation of generalized multiplication, it is found that the constant sequence $\left\{g \cdot f_n\right\}$, $f_n = f$, is equivalent to the sequence $\left\{g \cdot b_n\right\}$. Here $g \varepsilon C^\infty$ and $\left\{b_n\right\}$

is any sequence in g_t. Then the function $g \cdot f$ is embedded at $g \cdot g_t$ in G, if $g \varepsilon C^{\infty}$ and if f is embedded at g_t in G. In the above sense, the generalized multiplication in G is consistent with ordinary multiplication of a function f in B by a function g in C^{∞}.

In connection with differentiation, it is to be noted that f in B may be embedded at some function g_t in G but that $D'_x f$ may not be defined on the whole real line E_1, so that $D'_x f$ is not embedded in G. But $D'(g_t)$ always exists in G for any g_t in G, and sequences of normal functions always exist in $\widehat{D' g_t}$ that represent the generalized derivative of g_t.

CHAPTER III

SOME SEQUENCES IN THE EXTENSIONS OF THE GENERALIZED FUNCTIONS OF G

Several important types of sequences belonging to the union $\bigcup_{t\varepsilon T} \hat{g}_t$ of all the sequence families $\hat{g}_t$ corresponding to g_t will be considered here. As discussed previously, if any sequence $\{f_n\}$ belongs to $\bigcup_{t\varepsilon T} \hat{g}_t$ it must be a representation of a unique generalized function g_t in the set G.

Type I

For any b in B that has all orders of derivatives at the pre-selected fixed point a_1 used in the definition (2-1), it is always possible to find a sequence of functions in the set S_{a_1} that is equivalent to $\{b_n\}$, where $b_n = b$, (A-III, Case III). Therefore, if $b \varepsilon B$ and if b is infinitely differentiable at a_1, the sequence $\{b_n\}$, where $b_n = b$, is a sequence in $\bigcup_{t\varepsilon T} \hat{g}_t$. An example of a function in B that is infinitely differentiable at a_1 is

$$f(t) = f_o(t)[1 - u(t - t_1)] + f_1(t)[u(t - t_1) - u(t - t_2)] + \dots + f_k(t)\, u(t - t_k) \tag{3-1}$$

where $f_i(t) \varepsilon C^\infty$ for each $i = 0, \dots, k$, and where $t_1, t_2, \dots, t_k$ are not equal to a_1. In the expression for $f(t)$, $u(t - t_i)$ is the unit step function at $t = t_i$. A second example of a function in B that is infinitely differentiable at a_1 is

$$f(t) = \begin{cases} f_0(t) & , \quad t \in (t_j, t_j + T] \\ f(t \pm NT) & , \quad \text{for } N = 1, 2, \ldots \end{cases} \tag{3-2}$$

where $f_o(t) \in C^\infty$ and where $t_j \pm NT \neq a_1$ for any $N = 0, 1, 2, \ldots$. In this second example, $f(t)$ is a periodic function of period T.

<u>Type II</u>

A sequence in $\bigcup_{t\in T} \hat{g}_t$ that is not in S_{a_1} and is not constant is $\{f_n(t,t_j)\}$ where

$$f_n(t, t_j) = \begin{cases} 0 & , \quad t < t_j - \frac{1}{2n} \\ n & , \quad t_j - \frac{1}{2n} \leq t \leq t_j + \frac{1}{2n} \\ 0 & , \quad t_j + \frac{1}{2n} < t \end{cases} \tag{3-3}$$

and where t_j is any fixed point not equal to a_1, (A-III, case II).

For $a_1 < t_j$, $\{f_n\}$ has the property

$$\lim_{n\to\infty} \int_{a_1}^{t} f_n \, dx = \begin{cases} 0 & , \quad t < t_j \\ \frac{1}{2} & , \quad t = t_j \\ 1 & , \quad t > t_j \end{cases}$$

For any given n, f_n is shown in Figure 4.

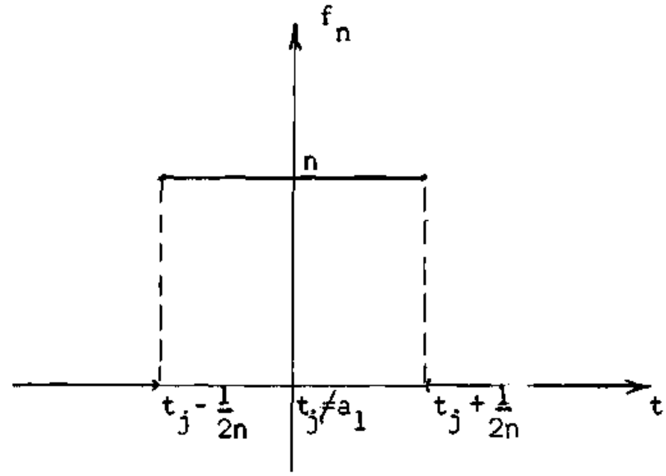

Figure 4. A Sequence of Unit Square Waves.

Type III

Define

$$S_n(t,t_j) = \begin{cases} \dfrac{n \exp\left[\dfrac{-1}{1 - n^2 (t-t_j)^2}\right]}{\displaystyle\int_{-\infty}^{\infty} \exp\left[\dfrac{-1}{1-x^2}\right] dx}, & |n(t-t_j)| \leq 1 \\ 0, & |n(t-t_j)| \geq 1 \end{cases} \tag{3-4}$$

where t_j is fixed in E_1 and $t_j \neq a_1$.

For each integer $i \geq 0$, $\left\{D_t^{(i)} S_n\right\}$ is a member of S_{a_1}, (A-III, Case I).

The sequence $\left\{S_n\right\}$ has the property

$$\lim_{n\to\infty}\int_{a_1}^{t} S_n\,dx = \begin{cases} 0\,, & t < t_j \\ \frac{1}{2}\,, & t = t_j \\ 1\,, & t_j < t \end{cases}$$

where $a_1 < t_j$.

The family of sequences in S_{a_1} equivalent to $\{S_n(t, t_j)\}$ is defined to be the delta function $\delta(t - t_j)$. The i^{th} generalized derivative of $\delta(t - t_j)$ is the unique generalized function equivalent to $\{D_t^{(i)} S_n(t, t_j)\}$. For each $i = 1, 2, \ldots,$ the function in G equivalent to $\{D_t^{(i)} S_n(t, t_j)\}$ is denoted by $\delta^{(i)}(t - t_j)$.

The sequence $\{S_n(t, t_j)\}$ is shown in Figure 5 for any given n.

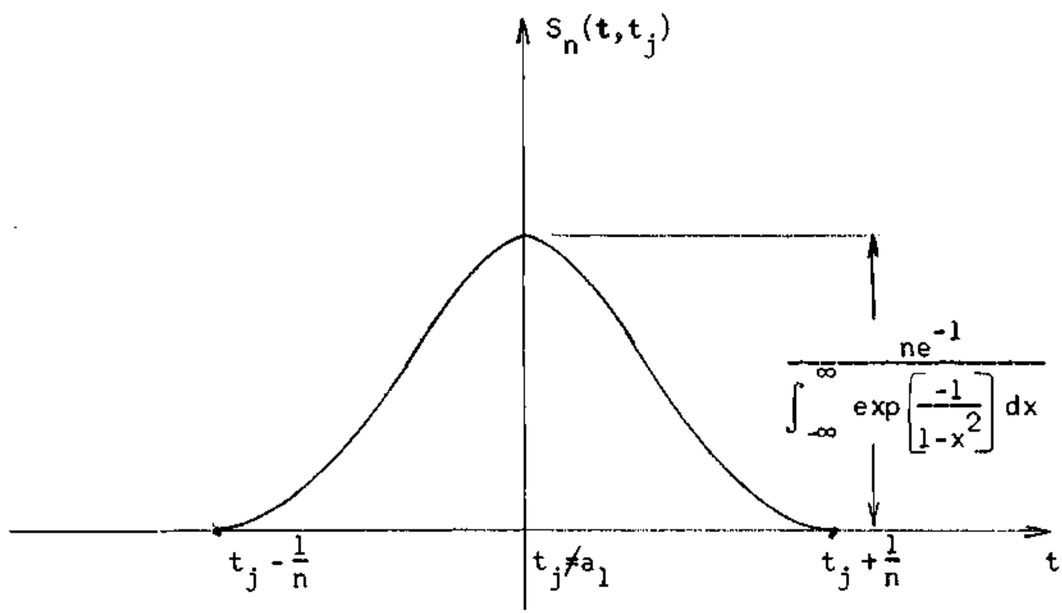

Figure 5. A Sequence of Normal Functions In $\delta(t - t_j)$.

The sequence $\{f_n(t, t_j)\}$ given above is equivalent to $\{S_n(t, t_j)\}$. That is $\{f_n(t, t_j)\} \varepsilon \hat{g}_t$ where $g_t = \delta(t - t_j)$. Since $\{f_n(t, t_j)\} \varepsilon \widehat{\delta(t - t_j)}$ this sequence is a representation for $\delta(t - t_j)$.

The sequence $\{p_n(t, t_j)\}$, where

$$p_n(t, t_j) = \begin{cases} 0 \;, & t < t_j - \frac{1}{n} \\ n^2 \;, & t_j - \frac{1}{n} \leq t \leq 0 \\ -n^2 \;, & 0 < t \leq t_j + \frac{1}{n} \\ 0 \;, & t_j + \frac{1}{n} < t \end{cases} \qquad (3\text{-}5)$$

with $t_j \neq a_1$, is a representation of the generalized function $\delta^{(1)}(t-t_j)$ since $\{S_n^{(1)}(t, t_j)\}$ is equivalent to $\{p_n(t, t_j)\}$. In the same manner, "step type" representations can be found for each of the generalized functions $\delta^{(i)}(t - t_j)$ where $t_j \neq a_1$, and $i = 2, 3, \ldots$.

The restriction $t_j \neq a_1$ in each of the above example sequences follows from equation (2-2) where divergence of a sequence of normal functions is not allowed at the point a_1.

CHAPTER IV

APPLICATION OF THE GENERALIZED MATHEMATICAL SYSTEM

The consistency of ordinary operations on the normal functions of B with the corresponding generalized operations on generalized functions in G was established in Chapter II. In many engineering problems, an ordinary operation such as differentiation of a normal function like the step function $u(t - t_j)$ fails to be defined. However, by the use of a generalized mathematical system, the result of the operation may be well defined. It is of importance to determine how the generalized mathematical system developed in Chapter II can be used in the event an ordinary operation on a normal function b in B is undefined. The usefulness of the mathematical system developed in Chapter II will be discussed in this chapter. In particular it will be shown that the step function $u(x - t_j)$ represents the generalized function in G that is the generalized integral of $\delta(x - t_j)$. Also, the generalized derivative of the generalized function in G represented by $u(x - t_j)$ is found to be the generalized function $\delta(x - t_j)$.

Suppose f denotes any of the ordinary operations of differentiation, integration, addition, or multiplication by a function of C^∞. If b is in B and is also embedded in G, a unique solution for the ordinary operation $f(b)$ is always defined in the following sense. If b is embedded at the generalized function g_t of G and if $f(b)$ is desired, the corresponding generalized operation $f(g_t)$ is always defined as a unique generalized function of G. Moreover, if the ordinary operation $f(b)$ is defined, the ordinary function $f(b)$ will be embedded in the generalized function $f(g_t)$.

The result of applying any of the four generalized operations defined for G to a member g_t of G is easily found. In each case, the generalized operation on a member of G is another unique generalized function in G. The result of any of the four generalized operations on g_t is the unique generalized function containing the sequence $\{f(b_n)\}$, where f is the ordinary operation corresponding to the generalized operation considered, and where $\{b_n\}$ is any sequence of normal functions in g_t.

For any sequence $\{f_n\}$ in the union $\bigcup_{t\varepsilon T} \hat{g}_t$, there is one and only one $\hat{g}_t$ containing $\{f_n\}$, and hence $\{f_n\}$ represents a unique generalized function g_t. If any of the generalized operations of differentiation, integration or multiplication by g of C^∞ are performed on g_t, there results a unique generalized function in G; $D'(g_t)$, $\int_{a_1}^{x}(g_t)$, or $g \cdot g_t$, respectively. For $D'(g_t)$, any sequence in the family $\widehat{D'(g_t)}$ is a representation of the generalized derivative of g_t. Similarly, any sequence in the family $\widehat{\int_{a_1}^{x}(g_t)}$ is a representation of the generalized integral of g_t. Also, any sequence in $\widehat{g \cdot g_t}$, where $g \varepsilon C^\infty$, represents the generalized product $g \cdot g_t$. Finally, any sequence in $\widehat{g_{t_1} + g_{t_2}}$ is a representation of the generalized sum of g_{t_1} and g_{t_2}.

If $\{f_n\} \varepsilon \hat{g}_{t_1}$ and $\{h_n\} \varepsilon \hat{g}_{t_2}$, the sequence $\{f_n + h_n\}$ will be in $\widehat{g_{t_1} + g_{t_2}}$ and hence will represent the generalized sum $g_{t_1} + g_{t_2}$.

The generalized operations in G are described in the following figures.

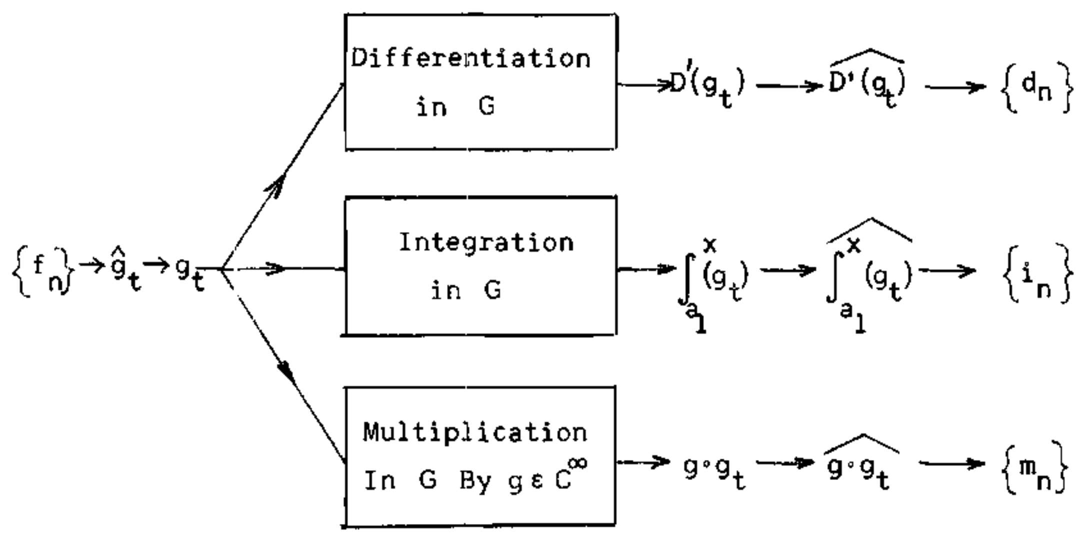

Figure 6. Differentiation, Integration, and Multiplication In G.

In Figure 6, $\{f_n\}$ is an arbitrary sequence in $\hat{g}_t$. As such, $\{f_n\}$ is an acceptable representation of the generalized function g_t in G. The sequence $\{d_n\}$ is an arbitrary sequence in $\widehat{D^{(1)}(g_t)}$ and hence is an acceptable representation of $D^{(1)}(g_t)$ in G. The sequence $\{i_n\}$ is an arbitrary sequence in $\widehat{\int_{a_1}^{x} (g_t)}$ and, is an acceptable

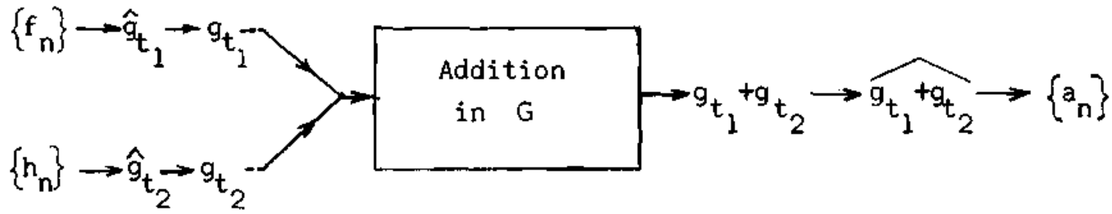

Figure 7. Addition In G.

representation of $\int_{a_1}^{x} (g_t)$. The sequence $\{m_n\}$ is an arbitrary sequence in $\widehat{g \cdot g_t}$ and hence is an acceptable representation of the generalized product $g \cdot g_t$, where $g \in C^{\infty}$.

In Figure 7, $\{f_n\}$ is an arbitrary sequence in $\hat{g}_{t_1}$ and $\{h_n\}$ is an arbitrary sequence in $\hat{g}_{t_2}$. Then $\{f_n\}$ and $\{h_n\}$ are acceptable representations for g_{t_1} and g_{t_2} in set G, respectively. The sequence $\{a_n\}$ is any sequence in $\widehat{g_{t_1} + g_{t_2}}$ and hence is an acceptable representation of $g_{t_1} + g_{t_2}$, the generalized sum of g_{t_1} and g_{t_2}.

If f is embedded in g_t in Figure 6, then $\{f_n\}$, where each $f_n = f$, is one sequence of $\hat{g}_t$ that represents g_t. The sequence $\{i_n\}$, where each $i_n = \int_{a_1}^{x} f\,dt$, is one acceptable representation for the generalized integral $\int_{a_1}^{x} (g_t)$ of the generalized function g_t. The sequence

$\{m_n\}$, where each $m_n = g \cdot f$, is one acceptable representation for the generalized product of g and g_t. Finally, if $D_x^{(1)}f$ is defined, the sequence $\{d_n\}$, where each $d_n = D_x' f$, is one acceptable representation for the generalized derivative $D^{(1)}(g_t)$ of the generalized function g_t.

If the normal function f is embedded in g_{t_1} and if h is a normal function embedded in g_{t_2} in Figure 7, the sequence $\{a_n\}$, where each $a_n = f + h$ is one acceptable representation for the generalized sum $g_{t_1} + g_{t_2}$.

To apply some of the above conclusions, consider the generalized function of G that contains the sequence $\{S_n(t, t_j)\}$ of equation (3-4), where a_1 is selected to be less than t_j. The generalized function in G containing $\{S_n(t, t_j)\}$ was defined to be the delta function $\delta(t - t_j)$ in Chapter III. Consider next the generalized integral $\int_{a_1}^{x} (\delta(t - t_j))$. The integral $\int_{a_1}^{x} (\delta(t - t_j))$ is that generalized function in G containing the sequence $\{f(x)\}$, where for each $n = 1, 2, \ldots,$

$$f_n(t) = \begin{cases} 0 & , \quad t \leq t_j - \frac{1}{n} \\ \int_{t_j - \frac{1}{n}}^{t} S_n(x,t_j)\, dt & , \quad t_j - \frac{1}{n} < t < t_j + \frac{1}{n} \\ 1 & , \quad t_j + \frac{1}{n} \leq t \end{cases} \qquad (4\text{-}1)$$

The sequence $\{f_n(x)\}$ is found to be equivalent to the constant sequence $\{h_n(x)\}$, where each $h_n(x) = u(x - t_j)$ for $n = 1, 2, 3, \ldots$. That is, the constant sequence $\{h_n\}$, where each $h_n = u(x - t_j)$, is one representation for the generalized integral of $\delta(x - t_j)$ in G. The results that apply to an embedded function can be made clearer with another simple example. Consider the step function $u(x - t_j)$ where t_j is fixed in E_1 and a_1 is selected such that $a_1 \neq t_j$. The function $u(x - t_j)$ is in B and it is embedded at some member of G. Let this member of G be g_t. The ordinary derivative of $u(x - t_j)$ is not defined at $x = t_j$, and hence $D_x^{(1)} u(x - t_j)$ is not defined on all E_1. However, the generalized derivative $D^{(1)}(g_t)$ does exist as a unique member of G, and there exist sequences of normal functions that represent $D^{(1)}(g_t)$. One such representation of $D^{(1)}(g_t)$ is found to be the sequence $\{S_n(x, t_j)\}$ discussed in Chapter III. The sequence $\{S_n(x, t_j)\}$ defines $\delta(t - t_j)$ in G. It can also be shown that another representative sequence for $D^{(1)}(g_t)$, where $u(x - t_j)$ is embedded in g_t, is the sequence $\{f_n(x, t_j)\}$ given by equation (3-3).

CHAPTER V

THE GENERALIZED SOLUTION IN G FOR A SYSTEM OF LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS

The behavior of many important electrical systems can be described by systems of linear constant coefficient differential equations, (l.c.c.d.e.). The response of a linear electrical system to some excitation of interest may not exist as a classical solution of the system of l.c.c.d.e.'s that describe the electrical system. For example the solution of a system of l.c.c.d.e.'s excited by impulse type functions is not defined in the sense of a classical system solution. However, it may be possible to determine the response of the electrical system to such an "improper" excitation by obtaining a generalized solution of the system of differential equations describing the electrical system.

The present chapter is devoted to the development of the generalized solution of a system of l.c.c.d.e.'s where the system of equations is excited by generalized functions in the set G.

The Non-homogeneous Equation In G

The linear nonhomogeneous differential equation of order m in G with constant coefficients is defined by $L_m(x) = g_t$. Here g_t belongs to G and $D^{(i)}$ is the notation for the generalized derivative of order (i) in G for the operator $L_m = D^{(m)} + p_1 D^{(m-1)} + \ldots + p_m$. For the development to follow the integer m is positive. That is, $m \geq 1$ will be considered.

A solution of $L_m(x) = g_t$ satisfying the set of boundary conditions $\xi_1,\ldots,\xi_m$ at $t = a_1$ is defined to be any element of G, say $g_{t_{n.h.}}$, which is equivalent to the sequence $\{\psi_n\}$ where $\{\psi_n\}$ is defined as follows. For any $\{b_n\} \varepsilon\, g_t$ and for each $n = 1, 2, \ldots,$ $\psi_n(t)$ is that unique solution of the ordinary nonhomogeneous equation $L_m(x) = b_n(t)$ with $\psi_n^{(i)}(a_1) = \xi_{i+1}$ for each $i = 0, 1,\ldots, m-1$, (App. IVA). For each n and each $\{b_n\} \varepsilon\, g_t$, $\psi_n(t)$ is uniquely determined by

$$\psi_n(t) = \psi_h(t) + \sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_{a_1}^{t} \frac{W_{k_o}(s)}{W(s)} b_n(s)\, ds \qquad (5\text{-}1)$$

where $\psi_h(t)$ is the unique solution of $L_m(x) = 0$ with $\psi_h^{(i)}(a_1) = \xi_{i+1}$ for each $i = 0,\ldots, m-1$; and where $\varphi_1,\ldots,\varphi_m$ is a fundamental set for $L_m(x) = 0$, (App. IVA).

For a sequence $\{b_n\} \varepsilon\, g_t$, the solution of $L_m(x) = g_t$ determined by $\{b_n\}$ is the element $g_{t_{n.h.}}$ containing the sequence

$$\{\psi_h(t)\} + \left\{ \sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_{a_1}^{t} \frac{W_{k_o}(s)}{W(s)} b_n(s)\, ds \right\} \qquad (5\text{-}2)$$

where $\psi_h^{(i)}(a_1) = \xi_{i+1}$ for each $i = 0, 1, \ldots m-1$. The equivalence of $\{b_n\}$ and $\{b_n^*\}$ in S_{a_1} implies the equivalence of the two sequences

$$\left\{ \sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_{a_1}^{t} \frac{W_{k_o}}{W} b_n\, ds \right\}$$

and

$$\left\{ \sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_{a_1}^{t} \frac{W_{k_o}}{W} b_n^* \, ds \right\}$$

in S_{a_1}. This implication follows directly from the generalized operations of multiplication, integration and addition defined for the set G, (App IVB). Then the solution $g_{t_{n.h.}}$ of $L_m(x) = g_t$ determined by $\{b_n\} \varepsilon\ g_t$ will be independent of the sequence $\{b_n\}$ selected from g_t. Therefore, the solution $g_{t_{n.h.}}$ in G determined by any given $g_t \varepsilon$ G will be the unique solution of $L_m(x) = g_t$ satisfying the prescribed boundary conditions $\xi_1, \ldots, \xi_m$ at $t = a_1$.

The Homogeneous Equation in G

Consider the solution of $L_m(x) = g_o$ satisfying the conditions $\xi_1, \ldots, \xi_m$ at $t = a_1$. The element g_o is that unique element of G that contains the constant sequence $\{0\}$. The unique solution $g_{t_{n.h.}}$ satisfying the given boundary conditions at a_1 must contain the sequence

$$\left\{ \Psi_h(t) \right\} + \left\{ \sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_{a_1}^{t} \frac{W_{k_o}}{W} f_n \, ds \right\} \tag{5-3}$$

where $\{f_n\} \varepsilon\ g_o$ and where $\Psi_n^{(i)}(a_1) = \xi_{i+1}$ for each $i = 0, \ldots, m-1$.

However, $\{f_n\} \varepsilon\ g_o$ implies that $\{f_n\} \sim \{0\}$ in S_{a_1}. Therefore,

$$\left\{ \sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_{a_1}^{t} \frac{W_{k_o}}{W} f_n \, ds \right\} \sim \left\{ 0 \right\} \text{ in } S_{a_1}$$

Then the solution $g_{t_{n.h.}}$ of $L_m(x) = g_o$ satisfying the boundary conditions ξ_{i+1} at a_1 for $i = 0,\ldots, m-1$ must be equivalent to the sequence $\{\psi_h(t)\} + \{0\}$. That is, $g_{t_{n.h.}}$ contains the constant sequence $\{\psi_h(t)\}$ where ψ_h is the unique solution of $L_m(x) = 0$ satisfying $\psi_h^{(i)}(a_1) = \xi_{i+1}$ for each $i = 0,.,\ldots,m-1$.

The equation $L_m(x) = g_o$ will be called the linear homogeneous equation of order m in G, and the solution of this equation satisfying the boundary conditions ξ_{i+1}, $i = 0,\ldots, m-1$ at $t = a_1$ will be the homogeneous solution of $L_m(x) = g_o$ with these prescribed boundary conditions.

The Nonhomogeneous System In G

The m^{th} order linear nonhomogeneous system of differential equations in G with constant coefficients is defined by

$$\begin{array}{l} M_{11}(x_1) + M_{12}(x_2)+\ldots+M_{1m}(x_m) = g_{t_1} \\ M_{21}(x_1) + M_{22}(x_2)+\ldots+M_{2m}(x_m) = g_{t_2} \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ M_{m1}(x_1) + M_{m2}(x_2)+\ldots+M_{mm}(x_m) = g_{t_m} \end{array} \tag{5-4}$$

where each $g_{t_i} \varepsilon G$ for $i = 1,\ldots, m$ and where each M is a generalized operator with constant coefficients.

A set of solutions for this system satisfying a prescribed set of boundary conditions $Q(a_1)$ at $t = a_1$ is defined to be a set of elements $g_1,\ldots,g_m \varepsilon G$ which are respectively equivalent to a set of sequences

$$\left\{\Psi_{1,n}\right\}, \ldots, \left\{\Psi_{m,n}\right\}$$

defined as follows. For $\left\{b_{1,n}\right\} \varepsilon\, g_{t_1}, \ldots, \left\{b_{m,n}\right\} \varepsilon\, g_{t_m}$ and for each fixed n, the set $\Psi_{1,n} \ldots, \Psi_{m,n}$ is a set of solutions of the ordinary nonhomogeneous system

$$\begin{array}{ccccccc} M_{11}(x_1) & + & M_{12}(x_2) & + \ldots + & M_{1m}(x_m) & = & b_{1,n}(t) \\ \vdots & & \vdots & & \vdots & & \vdots \\ M_{m1}(x_1) & + & M_{m2}(x_2) & + \ldots + & M_{mm}(x_m) & = & b_{m,n}(t) \end{array} \tag{5-5}$$

where the set of these solutions satisfy $Q(a_1)$ at $t = a_1$.

Every determinate system of linear differential equations with constant coefficients can be reduced to an equivalent diagonal system in which the dependent variables have any assigned diagonal order.[28] Then for the determinate system (5-5) there is the equivalent diagonal system

$$\begin{array}{ccccccc} N_{11}(x_1) + & N_{12}(x_2) & + \ldots + & N_{1m}(x_m) & = & f_{1n}(t) \\ & N_{22}(x_2) & + \ldots + & N_{2m}(x_m) & = & f_{2n}(t) \\ & & \ddots & \vdots & & \vdots \\ & & & N_{m,m}(x_m) & = & f_{m,n}(t) \end{array} \tag{5-6}$$

which has the set of solutions $\Psi_{1,n}, \ldots, \Psi_{m,n}$ satisfying $Q(a_1)$ at $t = a_1$.

The set $f_{1,n}, \ldots, f_{m,n}$ result from operations on $b_{1,n}, \ldots, b_{m,n}$ in the diagonalization of (5-5). Since $\left\{b_{i,n}\right\} \varepsilon\, S_{a_1}$ if $g_{t_i} \varepsilon\, G$ for $i = 1, \ldots, m$ and since $f_{i,n}(t)$ is some linear combination of derivatives of $b_{1,n}, \ldots, b_{m,n}$ for each n, then each $\left\{ f_{i,n} \right\} \varepsilon S_{a_1}$ for $i = 1, \ldots, m$. The equation $N_{m,m}(x_m) = f_{m,n}$ is some constant coefficient differential equation of some order, say 0_m.

Then this differential equation has a unique solution satisfying

$$\Psi^{(i)}_{m,n}(a_1) = \xi_{i+1,m}, \quad \text{where } i = 0, \ldots, (0_m - 1),$$

and it is given by

$$\Psi_{m,n}(t) = \Psi_{m,n,h}(t) + \sum_{k_o=1}^{0_m} \varphi_{k_o,m}(t) \int_{a_1}^{t} \frac{W_{k_o}}{W} f_{m,n}\, ds \quad (5\text{-}7)$$

where $\Psi_{m,n,h}$ is that solution of $N_{m,m}(x_m) = 0$ such that $\Psi^{(i)}_{m,n,h}(a_1) = \xi_{i+1,m}$ for each $i = 0, \ldots, (0_m - 1)$. Since

$$N_{m-1,m-1}(x_m) + N_{m-1,m}(x_m) = f_{m-1,n}(t), \quad (5\text{-}8)$$

the solution $\Psi_{m-1,n}(t)$ satisfying $\Psi^{(i)}_{m-1,n}(a_1) = \xi_{i+1,m-1}, i=0,\ldots,(0_{m-1}-1)$ is determined from

$$N_{m-1,m-1}(x_{m-1}) = f_{m-1,n}(t) - N_{m-1,m}(\Psi_{m,n}(t)) \quad (5\text{-}9)$$

where $\Psi^{(t)}_{m,n}$ is as given above.

In this fashion, the set of solutions $\Psi^{(t)}_{1,n}, \ldots, \Psi^{(t)}_{m,n}$ of (5-6) satisfying boundary conditions $Q(a_1)$ at $t = a_1$ can be found. The set of solutions $\Psi^{(t)}_{i,n}$, $i = 1, \ldots, m$ will be the set of solutions for (5-5) satisfying $Q(a_1)$ as well. Also this set of solutions will be unique for a sufficiently strong set of boundary conditions $Q(a_1)$ at $t = a_1$.

The requirements for the set of boundary conditions $Q(a_1)$ in order that a unique set of solutions of (5-5) be determined are those conditions needed for the determination of a unique solution of the normal homogeneous system.

$$\begin{aligned} M_{11}(x_1) + \dots + M_{1m}(x_m) &= 0 \\ M_{m1}(x_1) + \dots + M_{mm}(x_m) &= 0 \end{aligned} \tag{5-10}$$

If $\{b^*_{i,n}\} \sim \{b_{i,n}\}$ in S_{a_1} for $i = 1, \dots, m$ then it must follow that $\{f_{i,n}\} \sim \{f^*_{i,n}\}$ in S_{a_1} if each $f^*_{i,n}$ is formed from $b^*_{i,n}$ in the same way that $f_{i,n}$ is formed from $b_{i,n}$. Replacing $f_{i,n}$, $i = 1, \dots, m$ by $f^*_{i,n}$ in (5-6) results in a set $\Psi^*_{1,n}, \dots, \Psi^*_{m,n}$ of solutions satisfying $Q(a_1)$ at $t = a_1$. The solution $\Psi^*_{m,n}$ is given by

$$\Psi^*_{m,n}(t) = \Psi_{m,n,h}(t) + \sum_{k_o=1}^{O_m} \varphi_{k_{o,m}}(t) \int_{a_1}^{t} \frac{W_{k_o}}{W} f^*_{m,n}\, ds \tag{5-11}$$

where $\Psi_{m,n,h}(t)$ is the solution of $N_{m,m}(x_m) = 0$ with $\Psi^{(i)}_{m,n,h}(a_1) = \xi_{i+1,m}$ as before. Then the set $\Psi^*_{1,n}, \dots, \Psi^*_{m,n}$ is obtained from the set $\Psi_{1,n}, \dots, \Psi_{m,n}$ by replacing each $f_{i,n}$ by $f^*_{i,n}$ in the relations for $\Psi_{i,n}$ where $i = 1, \dots, m$. Therefore, since $\{f_{i,n}\} \sim \{f^*_{i,n}\}$ in S_{a_1}, it follows that

$$\{\Psi_{i,n}\} \sim \{\Psi^*_{i,n}\} \text{ in } S_{a_1} \tag{5-12}$$

Then for a given set $g_{t_1}, \dots, g_{t_m} \in G$ there is a unique set of generalized solutions $g_1, \ldots, g_m$ for the original system satisfying a sufficiently strong set of boundary conditions at $t = a_1$.

CHAPTER VI

SOLUTIONS OF THE GENERALIZED DIFFERENTIAL EQUATION $L_m(x) = g_t$

The solutions of $L_m(x) = g_t$ that correspond to some important members g_t in the set of generalized functions G are considered in this chapter. For the driving functions in G considered here, the solutions in G of $L_m(x) = g_t$ are investigated to determine whether they contain constant sequences. For any solution of $L_m(x) = g_t$ that does contain a constant sequence of normal functions, say $\{f_n\}$, where each $f_n = f$, the solution of $L_m(x) = g_t$ is normal in that the solution is represented by the normal function f. It will be shown that the solution of $L_m(x) = \delta(t - t_k)$, $t_k \neq a_1$, does contain a constant sequence of normal functions and hence it is a normal solution whenever the integer m is greater than zero. It will also be shown that some constant sequence formed from a normal function in B is always contained in the particular solution of $L_m(x) = g_t$ when any normal function b of Type I, chapter III is embedded in the generalized function g_t. Two important functions of Type I, Chapter III that are used in electrical engineering are defined by equations (3-1) and (3-2).

From equation (5-2), the unique solution $g_{t_{n.h.}}$ of $L_m(x) = g_t$ that satisfies the boundary conditions $\xi_1, \ldots, \xi_m$ at a_1 contains the sequence of S_{a_1}

$$\left\{ \Psi_h(t) \right\} + \left\{ \sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_{a_1}^{t} q_{k_o} b_n \, ds \right\} \tag{6-1}$$

where $\{b_n\} \varepsilon\, g_t$ and where $\Psi_h(t)$ is the unique solution of $L_m(x) = 0$ satisfying $\Psi_h^{(i)}(a_1) = \xi_{i+1}$ for $i = 0, \ldots, m-1$. For notation, q_{k_o} will be used for $\frac{W_{k_o}}{W}$.

The constant sequence $\{\Psi_h\}$ is independent of the sequence $\{b_n\}$ in S_{a_1} considered. Let

$$\{\Psi_{n,p}(t)\} = \left\{ \sum_{k_o=1}^{m} \varphi_{k_o} \int_{a_1}^{t} q_{k_o} b_n \, ds \right\}$$

The sequence $\{\Psi_{n,p}\}$ is a sequence of particular solutions for $L_m(x) = \{b_n\}$. From Chapter V, it is known that $\{\Psi_{n,p}\}$ is in S_{a_1}. For an arbitrary $\{b_n\}$ in S_{a_1}, the limit $\lim_{n\to\infty} \Psi_{n,p}$ may not exist. It is desired to know if $\lim_{n\to\infty} \Psi_{n,p}$ is in B for $\{b_n\}$ in S_{a_1}; and if this limit is in B does it follow that

$$\left\{ \lim_{n\to\infty} \Psi_{n,p} \right\} \sim \left\{ \Psi_{n,p} \right\}$$

For some important sequences $\{b_n\}$ in S_{a_1} it can be shown that $\left\{ \lim_{n\to\infty} \Psi_{n,p} \right\}$ is in B and is also equivalent to $\{\Psi_{n,p}\}$. However, it can also be shown that such is not the case for every sequence $\{b_n\}$ of S_{a_1}. Some examples will be given here.

Type 1

Consider $\left\{S_n(t,t_j)\right\}$ where $t_j \neq a_1$ as given in (3-4). The sequence of particular solutions $\left\{\Psi_{n,p}\right\}$ corresponding to $\left\{S_n\right\}$ is

$$\left\{ \sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_{a_1}^{t} q_{k_o} S_n \, ds \right\} \tag{6-2}$$

If $a_1 < t_j$,

$$\lim_{n\to\infty} \Psi_{n,p} = \begin{cases} 0 & , \quad t < t_j \\ \frac{1}{2} \sum_{k_o=1}^{m} \varphi_{k_o}(t_j)\, q_{k_o}(t_j) & , \quad t = t_j \\ \sum_{k_o=1}^{m} \varphi_{k_o}(t)\, q_{k_o}(t_j) & , \quad t > t_j \end{cases} \tag{6-3}$$

This limit is in B since $\varphi_{k_o} \varepsilon C^\infty$ for each $k_o = 1,\ldots, m$. For $a_1 < t_j$ the sequence of (6-2) is also equivalent to $\left\{\lim_{n\to\infty} \Psi_{n,p}\right\}$ where this limit is given in (6-3), (A-V).

Type 2

Consider the sequence of derivatives of S_n given by $\left\{S_n^{(i)}(t, t_j)\right\}$ where (i) is some positive integer and $a_1 < t_j$. The sequence $\left\{S_n\right\}$ is given by (3-4) as before. The sequence of particular solutions $\left\{\Psi_{n,p}\right\}$ corresponding to $\left\{S_n^i (t, t_j)\right\}$ is given by[(A-V)]

$$\Psi_{n,p} = \sum_{k_o=1}^{m} \sum_{r=o}^{i-1} (-1)^r \operatorname{Re}(\varphi_{k_o} q_{k_o}^{(r)}) \cdot S_n^{(i-r-1)}$$

$$+ \sum_{r=o}^{i-1} (-1)^{r+1} \operatorname{Re}(\varphi_{k_o}(t) \cdot q_{k_o}^{(r)}(a_1)) \cdot S_n^{(i-r-1)}(a_1, t_j)$$

$$+ (-1)^i \int_{a_1}^{t} \operatorname{Re}(\varphi_{k_o}(t) \cdot q_{k_o}^{(i)}(s)) \cdot S_n(s, t_j)\, ds$$

The sequence $\left\{\Psi_{n,p}\right\}$ corresponding to $\left\{S_n^{(i)}\right\}$ is equivalent to the sum of the two sequences

$$\left\{ \sum_{k_o=1}^{m} \sum_{r=o}^{i-1} (-1)^r \operatorname{Re}(\varphi_{k_o} \cdot q_{k_o}^{(r)}) \cdot S_n^{(i-r-1)} \right\}$$

and

$$\left\{ \sum_{k_o=1}^{m} (-1)^i H_{k_o}^{(i)} \right\}$$

where the second of these sequences is the constant sequence with

$$H_{k_o}^{(i)}(t) = \begin{cases} 0 & , \quad t < t_j \\ \frac{1}{2} \operatorname{Re}(\varphi_{k_o}(t_j) \cdot q_{k_o}^{(i)}(t_j)), & t = t_j \\ \operatorname{Re}(\varphi_{k_o}(t) \cdot q_{k_o}^{(i)}(t_j)) & , \quad t > t_j \end{cases}$$

That is, the sequence $\left\{\Psi_{n,p}\right\}$ for $\left\{S_n^{(i)}\right\}$ is equivalent to the sum of two sequences one of which is a <u>constant</u> sequence and the other a sequence that is a composite of $\left\{S_n^r(t, t_j)\right\}$ where $r = 0, \ldots, (i - 1)$.

It can be seen that $\lim_{n \to \infty} \Psi_{n,p}$ may or may not exist for the excitation sequence $\{S_n^i(t, t_j)\}$ where $i \geq 1$. Even when $\lim_{n \to \infty} \Psi_{n,p}$ exists in B it need not follow that $\{\Psi_{n,p}\}$ is equivalent to $\{\lim_{n \to \infty} \Psi_{n,p}\}$ since $\{S_n^r (t, t_j)\}$, $r = 0, \ldots, i - 1$, may be present as component sequences of $\{\Psi_{n,p}\}$.

<u>Type 3</u>

Consider the solution of $L_m(x) = g_t$ where g_t contains the sequence $\{b_n\}$ with $\{b_n\}$ as follows. Suppose $\{b_n\}$ of g_t is such that $b = \lim_{n \to \infty} b_n$ exists in B and $\{b_n\} \sim \{b_n\}$ where $\{b\}$ is a constant sequence with b as each term of the sequence. Also, let $\{b_n\}$ be a boundedly convergent sequence on each interval $[a_1, x_o]$ in E_1. For such a sequence $\{b_n\}$ the sequence of corresponding solutions $\{\Psi_{n,p}\}$ of $L_m(x) = b_n$ must be equivalent to the constant sequence $\{\lim_{n \to \infty} \Psi_{n,p}\}$ with $\lim_{n \to \infty} \Psi_{n,p}$ in B.

For any constant sequence $\{b\}$ of the Type I of Chapter III there exists a sequence $\{b_n\}$ in S_{a_1} with the characteristics

$$b = \lim_{n \to \infty} b_n ,$$

$$\{b_n\} \sim \{b\} \quad \text{and}$$

$\{b_n\}$ is boundedly convergent on each $[a_1, x_o]$ in E_1.

The following can be concluded for a function b of Type I given in Chapter III, where b is embedded at a unique g_t in G. For any sequence $\{b_n\}$ in g_t the particular generalized solution of

$L_m(x) = g_t$ contains the sequence of solutions $\{\Psi_{n,p}\}$ where $\Psi_{n,p}$ is the particular solution of $L_m(x) = b_n$ described previously in this chapter. The sequence $\{\Psi_{n,p}\}$ is in the particular generalized solution of $L_m(x) = g_t$ but also $\{\Psi_{n,p}\} \sim \{\lim_{n \to \infty} \Psi_{n,p}\}$ where $\lim_{n \to \infty} \Psi_{n,p}$ is a function in B. That is a constant sequence formed from a function of B is in the particular solution of $L_m(x) = g_t$ when a function b of Type I, Chapter III is embedded in g_t. Two important functions of Type I, Chapter III are defined by equations (3-1) and (3-2).

CHAPTER VII

EXAMPLES

Example 1

In Figure 8, $e_1 = \delta(t - t_1)$ in G and $e_2 = \delta(t - t_2)$ in G, where $t_1 > a_1$ and $t_2 > a_2$ are assumed. The normal functions f_1 and f_2 are in the set C^∞, and they multiply e_1 and e_2 respectively.

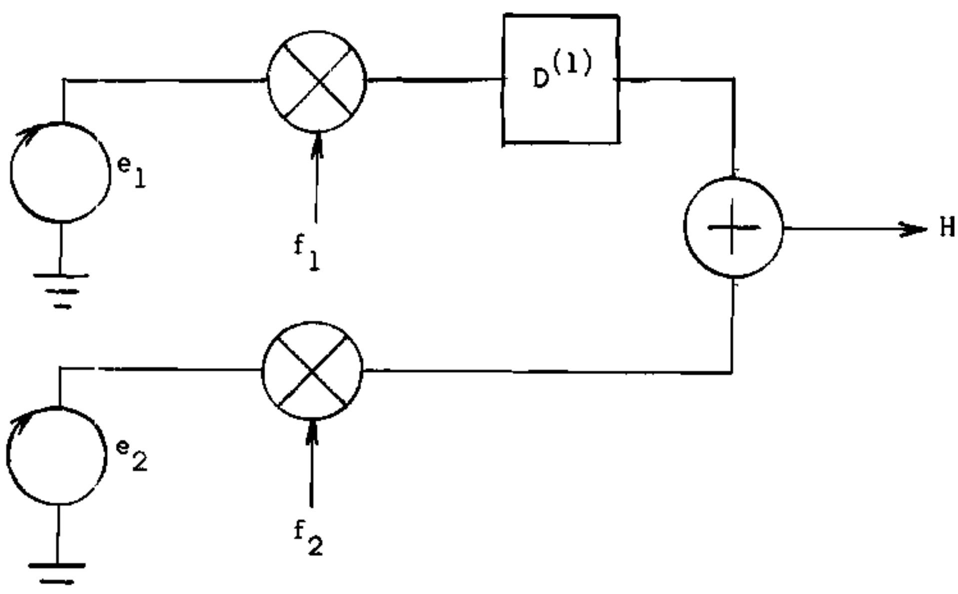

Figure 8. Multiplication and Differentiation Network.

The product of e_1 and f_1 is the unique member of G that contains the sequence

$$\left\{ f_1(t) \cdot S_n(t,\ t_1) \right\}$$

The product of e_2 and f_2 is the unique member of G that contains the sequence

$$\left\{ f_2(t) \cdot S_n(t,\ t_2) \right\}$$

where S_n is given by (3-3).

The derivative $D'(e_1 \cdot f_1)$ is the unique member of G that contains the sequence

$$\left\{ D'_t\ (f_1 \cdot S_n(t,\ t_1)) \right\} = \left\{ f_1 \cdot S'_n(t,\ t_1) + f'_1 \cdot S_n(t, t_1) \right\}$$

Then $D'(e_1 \cdot f_1)$ is the unique member of G that is equal to the sum of the unique members of G containing the sequences

$$\left\{ f_1(t) \cdot S'_n(t,\ t_1) \right\} \text{ and } \left\{ f'_1(t) \cdot S_n(t,\ t_1) \right\}$$

respectively. Therefore,

$$D^1(e_1 \cdot f_1) = f_1 \cdot \delta^1(t - t_1) + f_1^{\ 1} \cdot \delta(t - t_1)$$

From this result it follows that H is the unique member of G given by the sum

$$f_1 \cdot \delta^1(t - t_1) + f_1^{\ 1} \cdot \delta(t - t_1) + f_2 \cdot \delta(t - t_2)$$

This result, found by consistent analysis in the system G, is in agreement with the solution obtained using formal classical steps. Using formal operations gives

$$\frac{d}{dt}(e_1 f_1) = f_1(t) \cdot \delta^1(t - t_1) + f_1^1(t) \cdot \delta(t - t_1)$$

and

$$H = f_1 \cdot \delta^1(t - t_1) + f_1^1 \delta(t - t_1) + f_2 \delta(t - t_2)$$

Example II

The solution of $L_m(x) = \delta(t - t_j)$, $a_1 < t_j$, satisfying the m boundary conditions $\xi_1, \ldots, \xi_m$ at $t = a_1$ is the unique member function of G containing the sequence

$$\left\{\Psi_n(t)\right\} = \left\{\Psi_h(t)\right\} + \left\{\sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_{a_1}^{t} \frac{W_{k_o}}{W} S_n(s, t_j)\, ds\right\} \qquad (7\text{-}1)$$

where Ψ_h is the unique solution of $L_m(x) = 0$ satisfying $\Psi_h^{(i)}(a_1) = \xi_{i+1}$, $i = 0, \ldots, m-1$.

From Appendix V sequence type 1, the second sequence on the right in (7-1) is equivalent to the constant sequence

$$\left\{\lim_{n\to\infty} \Psi_{n,p}\right\} \qquad (7\text{-}2)$$

as given in equation (A5-2). Then the solution of $L_m(x) = \delta(t - t_j)$ satisfying $\xi_1, \ldots, \xi_m$ at a_1 is equivalent to the constant sequence

$$\left\{\Psi_h + \lim_{n\to\infty} \Psi_{n,p}\right\} \text{ where the} \qquad (7\text{-}3)$$

ordinary function composing this constant sequence is a member of B.

From (A5-2),

$$\lim_{n \to \infty} \Psi_{n,p} = \begin{cases} 0 & , \quad t < t_j \\ \frac{1}{2} \sum_{k_o=1}^{m} \varphi_{k_o}(t_j) \frac{W_{k_o}(t_j)}{W(t_j)} & , \quad t = t_j \\ \sum_{k_o=1}^{m} \varphi_{k_o}(t) \frac{W_{k_o}(t_j)}{W(t_j)} & , \quad t > t_j \end{cases} \tag{7-4}$$

From Appendix IV-A, $\Psi_h(t)$ is some linear combination of $\varphi_1, \ldots, \varphi_m$.

As a simple example of $L_m(x) = \delta(t - t_j)$, the RLC circuit of Figure 9 will be considered.

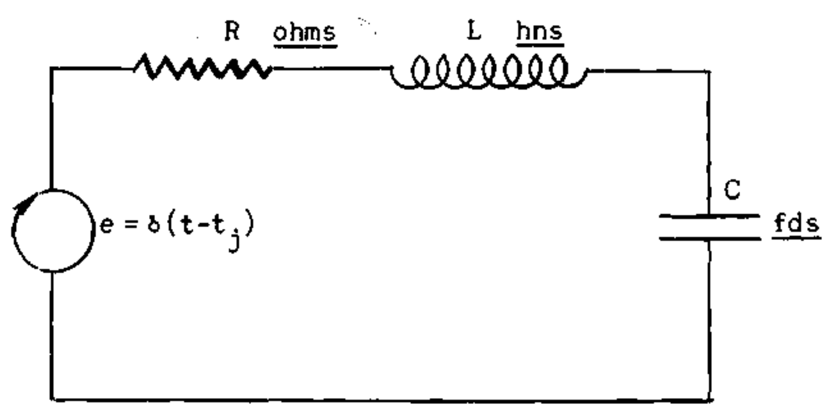

Figure 9. RLC Series Network.

$a_1 = 0$ is selected and $a_1 < t_j$ is considered. The circuit will be assumed to be initially at rest.

The charge $q(t)$ on C is described by

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC} q = \frac{1}{L}\delta(t - t_j) \tag{7-5}$$

and with zero initial conditions the homogeneous solution is the zero solution.

The solution in G of

$$\ddot{q} + p_1 \dot{q} + p_2 q = \delta(t - t_j) , \tag{7-6}$$

with p_1 and p_2 constants, where the homogeneous solution is zero, must be the member of G that is equivalent to the constant sequence $\left\{\lim_{n\to\infty} \Psi_{n,p}\right\}$ where $\lim_{n\to\infty} \Psi_{n,p}$ is given by (7-4) with $m = 2$.

A fundamental set of solutions for $\ddot{q} + p_1\dot{q} + p_2 q = 0$ is

$$\varphi_1 = e^{\lambda_1 t} \text{ and } \varphi_2 = e^{\lambda_2 t} \text{ if } \lambda_1 \neq \lambda_2$$

or
$$\varphi_1 = e^{\lambda_1 t} \text{ and } \varphi_2 = te^{\lambda_2 t} \text{ if } \lambda_1 = \lambda_2 ,$$

where λ_1 and λ_2 are solutions of the equation $r^2 + p_1 r + p_2 = 0$.

In the following analysis it will be assumed that $\lambda_1 \neq \lambda_2$, but the procedure to follow is unchanged if $\lambda_1 = \lambda_2$.

For the second order equation under consideration,

$$W(\varphi_1, \varphi_2) = \varphi_1\dot{\varphi}_2 - \dot{\varphi}_1\varphi_2$$

where $\dot{\varphi}_1 = \lambda_1 e^{\lambda_1 t}$ and $\dot{\varphi}_2 = \lambda_2 e^{\lambda_2 t}$. Then $W(\varphi_1, \varphi_2) = (\lambda_2 - \lambda_1)e^{(\lambda_2+\lambda_1)t}$. For $K_o = 1$ in (7-4), $W_1(\varphi_1, \varphi_2) = -e^{\lambda_2 t}$ and for $K_o = 2$, $W_2(\varphi_1, \varphi_2) = e^{\lambda_1 t}$. Then

$$\varphi_1(t)\,\frac{W_1(t_j)}{W(t_j)} = -\,\frac{e^{\lambda_1(t-t_j)}}{\lambda_2-\lambda_1}$$

and

$$\varphi_2(t)\,\frac{W_2(t_j)}{W(t_j)} = \frac{e^{\lambda_2(t-t_j)}}{\lambda_2-\lambda_1}$$

Therefore, the solution of (7-6) is the unique function of G equivalent to $\left\{\lim_{n\to\infty}\Psi_{n,p}\right\}$ where

$$\lim_{n\to\infty}\Psi_{n,p} = \begin{cases} 0 & , \quad t \le t_j \\ \dfrac{e^{\lambda_2(t-t_j)} - e^{\lambda_1(t-t_j)}}{\lambda_2-\lambda_1} & , \quad t > t_j \end{cases} \tag{7-7}$$

The dimensions are coulombs of charge.

The solution of $r^2 + p_1 r + p_2 = 0$ for λ_1 and λ_2 gives

$$\lambda_1 = -\frac{p_1}{2} + \frac{1}{2}\sqrt{p_1^2 - 4p_2}$$

and

$$\lambda_2 = -\frac{p_1}{2} - \frac{1}{2}\sqrt{p_1^2 - 4p_2}\ , \tag{7-8}$$

where $\lambda_1 \neq \lambda_2$ if $p_1^2 \neq 4p_2$.

If $p_1 = \frac{R}{L}$ and $p_2 = \frac{1}{LC}$, equation (7-7) represents the solution of (7-5) for zero initial conditions.

Using the conventional Laplace transform theory, the solution of (7-5) is

$$\frac{u(t-t_j)}{\lambda_2 - \lambda_1}\left(e^{\lambda_2(t-t_j)} - e^{\lambda_1(t-t_j)}\right) \text{ coulombs,} \tag{7-9}$$

which is the same as (7-7) except possibly at the point $t_j = t$.

It follows that the solution of (7-5) using the generalized procedure of this work is in agreement with the solution obtained by conventional methods of Laplace-transform theory.

Example III

In the circuit of Figure 10, e_1 is assumed to have all orders of ordinary derivatives for the moment. Also the circuit is at rest initially, when $t = 0$. The charge q_2 on C is to be found as a function of time.

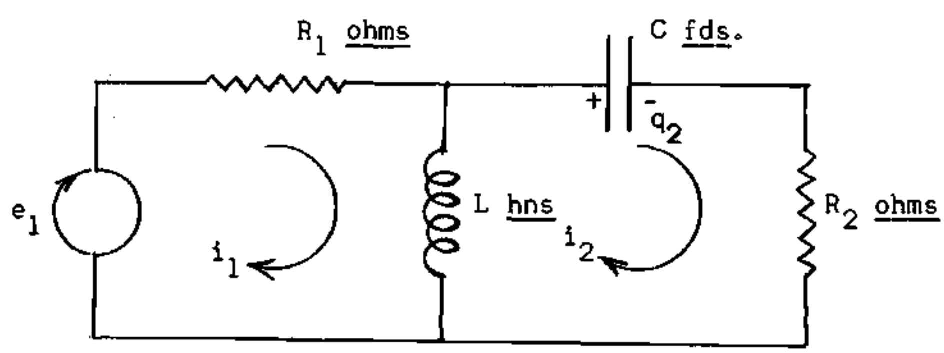

Figure 10. Two Mesh RLC Network.

The mesh equations in the currents i_1 and i_2 are

$$e_1 = R_1 i_1 + L\frac{d}{dt}(i_1 - i_2) \tag{7-10}$$

$$0 = L\frac{d}{dt}(i_2 - i_1) + \frac{1}{C}\int_0^t i_2\, dt + R_2\, i_2$$

with $\frac{dq_1}{dt} = i_1$ and $\frac{dq_2}{dt} = i_2$ the pair (7-10) become

$$e_1 = (LD^2 + R_1 D)q_1 - (LD^2)\, q_2 \tag{7-11}$$

$$0 = -\,(LD^2)\, q_1 + (LD^2 + R_2 D + \frac{1}{C})\, q_2$$

An equivalent set of equations obtained by diagonalization[29] of (7-11) is the pair

$$\left[\frac{(R_1 + R_2)}{L} D^2 + (\frac{1}{LC} + \frac{R_1 R_2}{L^2})\, D + \frac{R_1}{CL^2}\right] q_2 = \frac{D(e_1)}{L}$$

$$Dq_1 + \left[\frac{(R_1 + R_2)}{L} D^2 + \left(\frac{1}{LC} + \frac{R_1 R_2}{L^2} + \frac{R_2}{R_1}\right) D + \left(\frac{R_1}{CL^2} + \frac{1}{R_1 C}\right)\right] q_2 = \frac{1}{L}\,(D + \frac{L}{R_1})\, e_1 \tag{7-12}$$

For the particular citcuit values of $R_1 = R_2 = 1$ ohm, $L = 1$ henry, and $C = 1$ faried, (7-12) becomes the equation pair,

$$(2D^2 + 2D + 1)\, q_2 = De_1 \tag{7-13}$$

$$Dq_1 + (2D^2 + 3D + 2)\, q_2 = (D + 1)\, e_1$$

If the voltage e_1 is considered to be the generalized function $\delta(t - t_1)$, $t_1 > a_1$ which is in G, the solution for q_2 can be obtained for zero initial conditions at $t = 0$ by solving

$$\left(D^2 + D + \frac{1}{2}\right)\, q_2 = \frac{D(\delta(t - t_1))}{2} \tag{7-14}$$

The roots of the characteristic equation $r^2 + r + \frac{1}{2} = 0$ are

$$\lambda_1 = \frac{1}{2}(-1 + j) \quad \text{and} \quad \lambda_2 = \frac{1}{2}(-1 - j).$$

Then a fundamental set of solutions for the homogeneous equation of (7-14) is

$$\varphi_1 = e^{-\frac{1}{2}(1-j)t}, \quad \varphi_2 = e^{-\frac{1}{2}(1+j)t}$$

Since $e_1 = \delta(t - t_1)$, $t_1 > a_1$, e_1 is equivalent to $\{S_n(t, t_1)\}$ where S_n is given by (3-3). Therefore, with reference to (6-5), the solution in G of (7-14) satisfying zero initial conditions at $t = 0$ must be the unique member of G equivalent to the sequence

$$\left\{ \text{Re}\left(\varphi_1 g_1 + \varphi_2 g_2\right) \cdot \frac{S_n(t, t_1)}{2} \right\} + \tag{7-15}$$

$$\left\{ -\frac{1}{2}\left(H_1^{(1)}(t) + H_2^{(1)}(t)\right\} ,$$

where $g_1(t) = \dfrac{W_1(t)}{W(t)}$, $g_2 = \dfrac{W_2(t)}{W(t)}$; and where

$$H_1^{(1)}(t) + H_2^{(1)}(t) = U(t - t_1) \cdot \text{Re}\left[\varphi_1(t) \cdot g_1^{(1)}(t_1) + \varphi_2(t) g_2^{(1)}(t_1)\right]$$

With reference to the equations for W_1, W_2 and W given in Example II and noting that these expressions hold for equation (7-14),

$$g_1(t) = \frac{e^{-\lambda_1(t)}}{\lambda_1 - \lambda_2}, \quad g_2(t) = \frac{e^{-\lambda_2 t}}{\lambda_2 - \lambda_1}$$

Then $$\varphi_1(t)\cdot g_1(t)+\varphi_2(t)\cdot g_2(t) = \frac{e^{(\lambda_1-\lambda_1)t}}{\lambda_1-\lambda_2}+\frac{e^{(\lambda_2-\lambda_2)t}}{\lambda_2-\lambda_1} = 0$$

and

$$\varphi_1(t)\ g_1^{(1)}(t_1)+\varphi_2(t)\ g_2^{(1)}(t_1) = -\frac{\lambda_1}{\lambda_1-\lambda_2}\ e^{\lambda_1(t-t_1)}-\frac{\lambda_2}{\lambda_2-\lambda_1}\ e^{\lambda_2(t-t_1)}$$

Therefore, the solution g_2 in G of (7-14) must be equivalent to the constant sequence

$$\left\{-\frac{1}{2}\ u(t-t_1) \bullet \mathrm{Re}\left[\varphi_1(t)\ g_1^{(1)}(t_1)+\varphi_2(t)\ g_2^{(1)}(t_1)\right]\right\}$$

$$=\left\{\frac{1}{2}\ u(t-t_1)e^{-\frac{1}{2}(t-t_1)}\left[\cos\left(\frac{t-t_1}{2}\right)-\sin\left(\frac{t-t_1}{2}\right)\right]\right\} \qquad (7\text{-}16)$$

From Laplace transform theory, the solution for q_2 is

$$q_2 = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{se^{-st_1}}{s^2+s+\frac{1}{2}}\right] =$$

$$\frac{1}{2}\ e^{-\frac{1}{2}(t-t_1)}\left[\cos\left(\frac{t-t_1}{2}\right)-\sin\left(\frac{t-t_1}{2}\right)\right]u(t-t_1) \qquad (7\text{-}17)$$

It can be seen that the solution (7-17) agrees with the ordinary function in the constant sequence (7-16) which represents the generalized solution in G of the equation (7-14).

With the solution in G of (7-14) known, this solution can be substituted into (7-13) to solve for q_1 in G.

$$Dg_1 = (D+1)e_1 + (2D^2+3D+2)\ q_2$$

where $e = \delta(t-t_1)$, $t_1 > a_1$, and where q_2 is the unique member of G equivalent to the constant sequence (7-16).

CHAPTER VIII

CONCLUSIONS

The set G of generalized functions obtained in this work contains the improper impulse type functions, δ^k, $k = 0,1,2,\ldots;$ and most of the normal functions used in electrical engineering are embedded in G. The consistent operations of addition, differentiation, integration, and multiplication by functions in C^∞ are defined in G in a meaningful way. The generalized operations defined in G are consistent in that they are equivalent to the corresponding ordinary operations on any normal function embedded in G whenever the ordinary operations are defined for the normal function. The generalized operations defined in G are meaningful in that if one of these operations is performed on any given member of G, there always results a unique member in G. In the particular case of generalized addition of course, the operation is meaningful in that the sum of any two members of G is a unique member of G.

In many engineering problems, an ordinary operation such as differentiation of a normal function like the step function $u(t - t_j)$ fails to be defined in the ordinary sense. By application of the generalized mathematical system presented in the present work, such operations on normal functions may be well defined in the extended or generalized sense. In particular, it has been shown that the generalized derivative of the generalized function in G represented by $u(x - t_j)$ is the impulse function $\delta(x - t_j)$. Also, it is found that the step function $u(t-t_j)$ represents the generalized function in G that is the generalized integral of $\delta(x - t_j)$.

It has been shown that the representation of improper functions, such as $\delta(t)$ and its derivatives, by families of equivalent sequences of normal functions is indeed a fruitful representation. In the development of such a representation, the types of normal functions to consider for the set B are numerous. The selection of a set of sequences S from the set $\mathcal{B}$ of all possible infinite sequences formed from the selected B can also be performed in many ways. Having determined S, this set of sequences can be partitioned into disjoint subsets of equivalent sequences in various ways. Each of the disjoint subsets of sequences resulting from the partitioning of S can then be called a generalized function in a set of functions G.

In the particular development presented in the present work the selection of the set B, of the set S_{a_1}, and of the equivalence relation used to partition S_{a_1} are jointly determined. They are determined by the types of generalized functions desired in G, by the types of generalized operations to be defined in G, and by the fact that the operations defined in G are to be meaningful. In addition, the selection of B as a set of Riemann integrable functions was influenced by the desire to develop a generalized mathematical system based on mathematical theories that are familiar to the electrical engineer. The selection of equations (2-1), (2-2), and (2-3) in defining S_{a_1} and its partition were the result of experimentation, where the goal was to obtain a system containing important generalized functions and generalized operations.

By no means can it be concluded that the procedure of generalization presented in this work is the "best" procedure obtainable. Some other generalized system involving different sets B, S and a different partition of S might be better suited for some applications. In particular the extension to a greater number of generalized operations on the set of generalized functions might be easier in another system.

Each generalized function g_t in the set G is obtained in such a way that it corresponds uniquely to a set of sequences of normal functions $\hat{g}_t$, where in general g_t is contained in $\hat{g}_t$ but $\hat{g}_t$ is not contained in g_t. Then in general it can be said that a member g_t of G can be represented by sequences of normal functions other than those contained in g_t. Of course any sequence representing a given g_t in G must be equivalent under equation (2-3) to the sequences contained in g_t. In this manner the impulse and its derivatives can be represented by sequences of normal step type functions which is a desirable representation in electrical engineering work. However, such sequences of step type functions <u>are not</u> members of the families δ^k, $k = 0, 1, \ldots,$ in G. It is also found that most normal functions used in electrical engineering are representative of members of G when these normal functions are converted to constant sequences where every member of a given constant sequence is the same normal function.

The behavior of many important electrical systems can be described by systems of linear constant coefficient differential equations (l.c.c.d.e.). The response of a linear electrical system to some excitation of interest may not be defined as a classical solution of the system of l.c.c.d.e.'s describing the electrical system. For example, the solution of a system of l.c.c.d.e.'s excited by impulse type functions can not be determined by the classical solution of the system of equations. However, it may be possible to determine the system response to "improper" excitations by obtaining the generalized solution of the system of differential equations describing the electrical system.

The methods for solving linear constant coefficient differential equations (l.c.c.d.e.) and systems of l.c.c.d.e.'s in G have been presented. In each case the solution in G is found to be a unique member, or members, of G in the same sense that the ordinary solution of a l.c.c.d.e. or system of l.c.c.d.e.'s is a unique solution when sufficient boundary conditions are applied. In particular, for the m boundary conditions $\xi_1, \ldots, \xi_m$ at $t = a_1$, there is a unique solution in G for an m^{th} order l.c.c.d.e. that is excited by a function, such as δ, in G.

The solution of $L_m(x) = g_t$ in G has been determined when g_t is any one of several important member functions of G. From the cases considered for g_t, it can be concluded that the solution of $L_m(x) = g_t$ in G can have an embedded normal function when no normal function is embedded in g_t. In particular, it is found that the solution in G of $L_m(x) = \delta(t - t_k)$, where $t_k \neq a_1$, contains a constant sequence of normal functions, and therefore the solution in G of this differential equation can be considered to be a normal solution whenever $m > 0$.

Several examples are given for the application of the generalized mathematical system to electrical circuits. The solutions of these problems using the generalized system G are found to be consistent with the solutions obtained by the classical theory of Laplace-transforms.

Finally, it is concluded that the generalized mathematical system developed in this work has considerable potential intuitive content for the electrical engineer. This conclusion follows since the generalized functions of the set G are composed directly from sequences of normal functions, and the generalized operations defined on G are intimately related to the corresponding ordinary operations on the normal functions of any sequence representing a generalized function in G.

APPENDIX I

THE EXPANSION OF THE SEQUENCE $\left\{f^{k}_{x_1}(gb_n)\right\}$ WHERE K IS A POSITIVE INTEGER

With integration by Parts (A-II),

$$f^1\left(g^{k_1}\cdot f^{k_1}(b_n)\right) = g^{k_1}\cdot f^{k_1+1}(b_n) - f^1\left(g^{k_1+1}\cdot f^{k_1+1}(b_n)\right) \qquad \text{(A1-1)}$$

where $k_1 \geq 0$ is an integer. Using (A1-1), $f^1(gb_n)$ can be expanded into an arbitrary but finite number of terms. This expansion into k_1+1 terms is as follows.

$$f^1(gb_n) = g\cdot f^1(b_n)+\ldots+(-1)^{k_1+1}\, g^{k_1-1}\cdot f^{k_1}(b_n) + (-1)^{k_1}\cdot f^1(g^{k_1}\cdot f^{k_1}(b_n)) \qquad \text{(A1-2)}$$

Then $f^2(gb_n)$ has an expansion with k_1+1 terms using (A1-2) for $f^1(gb_n)$. For each $1 \leq j_1 \leq k_1$, the $j_1^{\underline{th}}$ term of this expansion of $f^2(gb_n)$ is

$$(-1)^{j_1+1}\cdot f^1\left(g^{j_1-1}\cdot f^{j_1}(b_n)\right) \qquad \text{(A1-3)}$$

and the $k_1+1^{\underline{th}}$ term is

$$(-1)^{k_1}\cdot f^2\left(g^{k_1}\cdot f^{k_1}(b_n)\right) \qquad \text{(A1-4)}$$

For any j_1 such that $1 \leq j_1 \leq k_1$, (A1-3) can be expanded into an arbitrary number of terms in the fashion of (A1-2). Then for each

$1 \le j_1 \le k_1$, let (A1-3) be expanded into $(k_{j1} + 1)$ terms with each $k_{j_1} \ge 1$ and arbitrary. For $1 \le j_2 \le k_{j_1}$, the $j_2^{\underline{th}}$ term of this expansion is

$$(-1)^{j_1+1} \cdot (-1)^{j_2+1} \cdot g^{j_1-1+j_2-1} \cdot f^{j_1+j_2}(b_n^2) \qquad \text{(A1-5)}$$

and the $(k_{j_1} + 1)^{\underline{th}}$ term of the expansion of (A1-3) is

$$(-1)^{j_1+1} \cdot (-1)^{k_{j_1}} \cdot f^1\left(g^{k_{j_1}+j_1-1} \cdot f^{k_{j_1}+j_1}(b_n)\right) \qquad \text{(A1-6)}$$

In light of the above two-stage expansion of $f^2(gb_n)$, a corresponding expansion of $f^3(gb_n)$ is obtained by noting that $f^3(gb_n) = f^1(f^2(gb_n))$. There are three types of terms in the expansion of $f^3(gb_n)$. There is the term

$$(-1)^{k_1} \cdot f^3\left(g^{k_1} \cdot f^k(b_n)\right) \qquad \text{(A1-7)}$$

resulting from (A1-4). There are the terms

$$(-1)^{j_1+1} \cdot (-1)^{k_{j_1}} \cdot f^2\left(g^{k_{j_1}+j_1-1} \cdot f^{k_{j_1}+j_1}(b_n)\right) \qquad \text{A1-8)}$$

where $1 \le j_1 \le k_1$. These follow from (A1-6). Finally, from (A1-5) there are the terms of the form

$$(-1)^{j_1+1} \cdot (-1)^{j_2+1} \cdot f^1\left(g^{j_1-1+j_2-1} \cdot f^{j_1+j_2}(b_n^2)\right) \qquad \text{(A1-9)}$$

where $1 \le j_1 \le k_1$ and $1 \le j_2 \le k_{j_1}$.

Each term of the form of (A1-9) can be expanded, in the fashion of

(A1-2), into $k_{j_2}+1$ terms, where $k_{j_2} \geq 1$ is an integer and is arbitrary for each $1 \leq j_2 \leq k_{j_1}$ with $1 \leq j_1 \leq k_1$.

For $1 \leq j_3 \leq k_{j_2}$, the $j_3^{\underline{th}}$ term of such an expansion must be

$$(-1)^{j_1+1} \circ (-1)^{j_2+1} \circ (-1)^{j_3+1} \circ g^{(j_1-1+j_2-1+j_3-1)} \circ f^{j_1+j_2+j_3}(b_n) \quad \text{(A1-10)}$$

and the $(k_{j_2}+1)^{\underline{th}}$ term in this expansion must be

$$(-1)^{j_1+1} \circ (-1)^{j_2+1} \circ (-1)^{k_{j_2}} \circ f^1\left(g^{k_{j_2}+j_2+j_1-2} \circ f^{k_{j_2}+j_2+j_1}(b_n)\right) \quad \text{(A1-11)}$$

If the above type expansion is developed for integers $k > 3$, it is found that $f^k(gb_n)$ has an expansion with terms of the following types.

$$(-1)^{k_1} \circ f^k(g^{k_1} \circ f^{k_1}(b_n)) \quad \text{(A1-12)}$$

where $k_1 \geq 0$ and is arbitrary.

$$(-1)^{k_{j_1}+j_1+1} \circ f^{k-1}\left(g^{k_{j_1}+j_1-1} \circ f^{k_{j_1}+j_1}(b_n)\right) \quad \text{(A1-13)}$$

where $1 \leq j_1 \leq k_1$ and $k_{j_1} \geq 1$ and arbitrary for each j_1.

$$(-1)^{k_{j_2}+j_2+j_1+2} \circ f^{k-2}\left(g^{k_{j_2}+j_2+j_1-2} \circ f^{k_{j_2}+j_2+j_1}\right) \quad \text{(A1-14)}$$

where $1 \leq j_1 \leq k_1$, $1 \leq j_2 \leq k_{j_1}$ and where $k_{j_2} \geq 1$ is arbitrary for each j_2.

For integers q such that $2 \leq q \leq k-1$ there are the terms

$$(-1)^{k_{j_q}+j_q+\cdots+j_1+q} \circ f^{k-q}\left(g^{k_{j_q}+j_q+\cdots+j_1-q} \circ f^{k_{j_q}+j_q+\cdots+j_1}(b_n)\right) \quad \text{(A1-15)}$$

where $1 \leq j_1 \leq k_1$, $1 \leq j_2 \leq k_{j_1}, \ldots, 1 \leq j_q \leq k_{j_{q-1}}$ with $k_{j_q} \geq 1$ an arbitrary integer for each j_q.

Finally, there are the expansion terms,

$$(-1)^{k + \sum_{i=1}^{k}(j_i)} \cdot g^{-k + \sum_{i=1}^{k}(j_i)} \cdot f^{\sum_{i=1}^{k}(j_i)}(b_n) \qquad \text{(A1-16)}$$

where $1 \leq j_1 \leq k_1$, $1 \leq j_2 \leq k_{j_1}, \ldots, 1 \leq j_{k-1} \leq k_{j_{k-2}}$, $1 \leq j_k \leq k_{j_{k-1}}$ with each $k_{j_{k-1}} \geq 1$ an arbitrary integer for each j_{k-1} such that $1 \leq j_{k-1} \leq k_{j_{k-2}}$.

As $n \to \infty$ the limit of (A1-16) is

$$(-1)^{k + \sum_{i=1}^{k}(j_i)} \cdot g^{-k + \sum_{i=1}^{k}(j_i)} \cdot \lim_{n \to \infty} f_{x_1}^{\sum_{i=1}^{k}(j_i)}(b_n) \qquad \text{(A1-17)}$$

where $x_1 \varepsilon E_1$. Then since $\{b_n\} \varepsilon S_{a_1}$ by assumption, it must follow that the limit (A1-17) exists in E_1 for all x_1 except possibly a finite number of points different from a_1 in each finite interval $[a_1, x_0]$ of E_1.

For each q such that $2 \leq q \leq k - 1$ in equation (A1-15), let k_{j_q} be selected such that the factor $k_{j_q} + \sum_{i=1}^{q}(j_i)$ exceeds the positive integer k_{b_n}, where k_{b_n} is defined in equation (2-2) for a sequence $\{b_n\}$ in S_{a_1}. It is noted in respect to the above selection of k_{j_q}, $2 \leq q \leq k - 1$, that $j_i \geq 1$ for each $i = 1, \ldots, q$. With the selection of k_{j_q} so made, it follows from Theorems IV, V and VI of Appendix II, that the limit of (A1-15) as $n \to \infty$ must equal

$$(-1)^{q + k_{j_q} + \sum_{i=1}^{q} (j_i)} \cdot f^{k-q}\left(g^{k_{j_q} - q + \sum_{i=1}^{q} (j_i)} \cdot \lim_{n \to \infty} f^{k_{j_q} + \sum_{i=1}^{q} (j_i)}\right) \tag{A1-18}$$

for each q, $2 \leq q \leq k - 1$.

Similarly, if k_1 is taken greater than or equal to k_{b_n}, the limit as $n \to \infty$ of equation (A1-12) must equal

$$(-1)^{k_1} \cdot f^k(g^{k_1} \cdot \lim_{n \to \infty} f^{k_1} (b_n)) \tag{A1-19}$$

If $k_{j_1} + j_1 \geq k_{b_n}$, the limit is $n \to \infty$ of (A1-13) must equal

$$(-1)^{k_{j_1} + j_1 + 1} \cdot f^{k-1}\left(g^{k_{j_1} + j_1 - 1} \cdot \lim_{n \to \infty} f^{k_{j_1} + j_1}(b_n)\right) \tag{A1-20}$$

where $1 \leq j_1 \leq k_1$.

For each integer $k \geq 1$, it follows that $f^k(gb_n)$ can be expanded into the sum of a finite number of terms of the types (A1-12) through (A1-16). Then it can be said in light of equations (A1-17) through (A1-20) that if $k \geq 1$ and if $g(x) \in C^{\infty}$,

$$\lim_{n \to \infty} f^k_{x_1} (gb_n) \tag{A1-21}$$

exists as a real number for every $x_1 \in E_1$ except for a possible finite number of points not equal to a_1 in each finite interval $[a_1, x_o]$ in E_1.

Suppose $k = k_{b_n}$. Then in (A1-16) $\sum_{i=1}^{k} (j_i) = \sum_{i=1}^{k_{b_n}} (j_i) \geq k_{b_n}$ since

$j_i \geq 1$ for each i. That is, the order of integration $\sum_{i=1}^{k}(j_i)$ for $f^{\sum_{i=1}^{k}(j_i)}(b_n)$ is at least as great as k_{b_n}. Therefore, since $\left\{f^{k_{b_n}}(b_n)\right\}$ is boundedly convergent and $\lim_{n\to\infty} f^{k_{b_n}}(b_n)$ is Riemann integrable on each $[a_1, x_o]$ in E_1, it must follow that the sequence

$$\left\{ g^{-k+\sum_{i=1}^{k}(j_i)} \cdot f^{\sum_{i=1}^{k}(j_i)}(b_n) \right\} \tag{A1-22}$$

is boundedly convergent and

$$\lim_{n\to\infty} g(x_1)^{-k+\sum_{i=1}^{k}(j_i)} \cdot f^{\sum_{i=1}^{k}(j_i)}(b_n)$$

is Riemann integrable on each finite interval $[a_1, x_o]$ in E_1 (Theorem V, A-II).

In equation (A1-15), if $k_{j_q} + \sum_{i=1}^{q}(j_i)$ is selected $\geq k_{b_n}$ for each q such that $0 \leq q \leq k - 1$ where $1 \leq j_i \leq k_{j_{i-1}}$ for each $i = 1,\ldots,q$; then with $k = k_{b_n}$ the sequence

$$\left\{ f^{k_{b_n}-q} \left(g^{k_{j_q}-q+\sum_{i=1}^{q}(j_i)} \cdot f^{k_{j_q}+\sum_{i=1}^{q}(j_i)}(b_n) \right) \right\} \tag{A1-23}$$

is boundedly convergent and the limit as $n \to \infty$ for this sequence is Riemann integrable on each finite interval $[a_1, x_o]$ in E_1.

Therefore, since the finite sum of boundedly convergent sequences is boundedly convergent, and since the limit of a sum is the sum of the limits when all limits exist, it must follow that the sequence $\left\{f^{k_{b_n}}(gb_n)\right\}$ is boundedly convergent and $\lim_{n\to\infty} f^{k_{b_n}}(gb_n)$ exists and is Riemann integrable on each finite interval $[a_1, x_o]$ in E_1.

Then $\left\{gb_n\right\} \varepsilon\ S_{a_1}$ whenever $\left\{b_n\right\} \varepsilon\ S_{a_1}$ if $g\ \varepsilon\ C^{\infty}$.

Consider next equivalent sequences $\left\{b_n\right\} \sim \left\{b_n^*\right\}$ in S_{a_1}. The forms of (A1-15) and (A1-16) are independent of which sequence of S_{a_1} is used in the development of these equations. Thus their form for b_n^* would be the same as for b_n. With reference to equation (2-2) it is seen that $k_{b_n^*}$ need not be equal to k_{b_n}. Then for $k \geq 1$ let $f^k(gb_n)$ and $f^k(gb_n^*)$ be developed in finite expansions of identical form where k_{j_q} is selected for equation (A1-15) such that $k_{j_q} + \sum_{i=1}^{q}(j_i)$ is greater than or equal to the maximum of k_{b_n} and $k_{b_n^*}$ for each q, $0 \leq q \leq k - 1$. Here, as before, $1 \leq j_i \leq k_{j_{i-1}}$ for each $1 \leq i \leq q$.

With the same expansion used for both $f^k(gb_n)$ and $f^k(gb_n^*)$ it is found that the terms of the two expansions have a one-to-one correspondence. In particular, for the terms like (A1-16) it is found that

$$\lim_{n\to\infty}\left| g^{-k+\sum_{i=1}^{k}(j_i)}(x_1) \cdot \left(f_{x_1}^{\sum_{i=1}^{k}(j_i)}(b_n) - f^{\sum_{i=1}^{k}(j_i)}(b_n^*)\right)\right| =$$

$$\left| g^{-k+\sum_{i=1}^{k}(j_i)}(x_1)\right| \cdot \lim_{n\to\infty}\left| f_{x_1}^{\sum_{i=1}^{k}(j_i)}(b_n) - f_{x_1}^{\sum_{i=1}^{k}(j_i)}(b_n^*)\right| = 0 \qquad \text{(A1-24)}$$

for all but possibly a finite number of points not equal to a_1 in each finite $[a_1, x_o]$ in E_1.

For terms like (A1-15) it is found that

$$\lim_{n\to\infty} \left| f_{x_1}^{(k-q)} \left(g^{k_{j_q} - q + \sum_{i=1}^{q} (j_i)} \cdot \left[f^{k_{j_q} + \sum_{i=1}^{q} (j_i)} (b_n) - f^{k_{j_q} + \sum_{i=1}^{q} (j_i)} (b_n^*) \right] \right) \right| = 0 \tag{A1-25}$$

for all x_1 in E_1 except possibly a finite number of points not equal to a_1 in each finite $[a_1, x_o]$ in E_1.

These results imply that if $k \geq 1$,

$$\lim_{n\to\infty} \left| f_{x_1}^k (gb_n) - f_{x_1}^k (gb_n^*) \right| = 0 \tag{A1-26}$$

for all but possibly a finite number of points not equal to a_1 in each finite $[a_1, x_o]$ in E_1.

With reference to (2-14), equation (A1-26) is also valid for all $k \leq 0$.

Therefore, if $g \in C^\infty$ and if $\{b_n\} \sim \{b_n^*\}$ in S_{a_1}, it follows that $\{gb_n\} \sim \{gb_n^*\}$ in set S_{a_1}.

APPENDIX II

SOME IMPORTANT THEOREMS OF MATHEMATICAL ANALYSIS

Theorem I. Integration by Parts[30]

If $f(x)$ is Riemann-Stieltjes integrable with respect to the integrator $\alpha(x)$ on $[a,b]$, then $\alpha(x)$ is Riemann-Stieltjes integrable on $[a,b]$ with respect to the integrator $f(x)$ and

$$\int_a^b f(x)\, d\alpha(x) + \int_a^b \alpha(x)\, df(x) = f(x)\, \alpha(x) \Big|_a^b$$

Theorem II. Reduction Theorem[31]

If $f(x)$ is Riemann-Stieltjes integrable with respect to $\alpha(x)$ on $[a, b]$ and if $\alpha(x)$ has a continuous derivative $\alpha^{(1)}(x)$ on $[a, b]$, then $\int_a^b f(x)\, \alpha^{(1)}(x)\, dx$ exists and is equal to $\int_a^b f(x)\, d\alpha(x)$.

From (II) it follows that if $g(x)$ and $b_n(x)$ are in C^∞,

$$\int_{a_1}^{x_1} g b_n \, dx_2 = \int_{a_1}^{x_1} g d\left[\int_{a_1}^{x_2} b_n \, dx_3\right]$$

if $\frac{d}{dx_2} \int_{a_1}^{x_2} b_n \, dx_3 = b_n(x_2)$. That this is the case follows from (III).

Theorem III. The Integral as a Function of the Interval[32]

If $f(x)$ is Riemann integrable on $[a, b]$ and if $F(x)$ is defined by

$$F(x) = \int_a^x f \, dt \quad \text{for} \quad x \in [a, b] \quad \text{then}$$

$F^{(1)}(x)$ exists at every point of $[a, b]$ where $f(t)$ is continuous and

$F^{(1)}(x)$ is given by $F^{(1)}(x) = f(x)$.

With $b_n(x) \varepsilon C^\infty$, $\frac{d}{dx_2} \int_{a_1}^{x_2} b_n dx_3 = b_n(x_2)$ for all $x_2 \varepsilon [a_1, x_1]$.
Then $f^1(gb_n) = \int_{a_1}^{x_1} g d\left[\int_{a_1}^{x_2} b_n dx_3\right]$. But by Theorem (I) $f^1(gb_n)$ must also equal

$$g(x_2) \cdot \int_{a_1}^{x_2} b_n dx_3 \Big]_{a_1}^{x_1} - \int_{a_1}^{x_1} g^{(1)} \cdot \int_{a_1}^{x_2} b_n dx_3 dx_2 .$$

Then $f^1(gb_n) = g \cdot f^1(b_n) - f^1(g^{(1)} \cdot f^1(b_n))$.

The equation (Al-1) is the result of Theorems I, II and III given here.

<u>Definition (I)</u>

A sequence of functions $\{f_n(x)\}$ is said to be uniformly bounded on an interval $[a, b]$ if there exist a constant $M > 0$ such that $|f_n(x)| \leq M$ for all x in $[a, b]$ and all $n = 1,2,3,\ldots$. The positive number M is called a uniform bound for $\{f_n(x)\}$.

<u>Definition (II)</u>

A sequence of functions $\{f_n(x)\}$ is said to be boundedly convergent on an interval $[a, b]$ if $\{f_n(x)\}$ is convergent pointwise in $[a, b]$ and uniformly bounded on $[a, b]$.

<u>Theorem IV. Arzelà's Theorem for Boundedly Convergent Sequences</u>[33]

If a sequence $\{f_n(x)\}$ is boundedly convergent on an interval $[a, b]$ in E_1, and if each $f_n(x)$ as well as $\lim_{n\to\infty} f_n(x)$ is Riemann-integrable on $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b f_n dx = \int_a^b \lim_{n\to\infty} f_n dx = \int_a^b f dx .$$

Theorem V

Let $g(x) \varepsilon C^{\infty}$ and suppose $\{f_n(x)\}$ is boundedly convergent on $[a, b]$ with each f_n and $\lim_{n \to \infty} f_n$ Riemann integrable on $[a, b]$. Then the sequence $\{gf_n\}$ is boundedly convergent on $[a, b]$ and each gf_n and $\lim_{n \to \infty} gf_n$ is Riemann integrable on $[a, b]$. Theorem (V) follows below. Since $\lim_{n \to \infty} gf_n = g \cdot f$ and since the product of two Riemann integrable functions is Riemann-integrable, each gf_n as well as $g \cdot f$ are Riemann integrable on $[a, b]$. Since $\{f_n\}$ converges pointwise at each x in $[a, b]$, so also must the sequence $\{gf_n\}$. Since $\{f_n\}$ is uniformly bounded on $[a, b]$, there is a positive number M such that $|f_n| \leq M$ for all $x \varepsilon [a, b]$ and $n = 1,2,3,\ldots$. But $|f_n g| = |f_n||g| \leq M \cdot |g|$ for all $x \varepsilon [a, b]$ and all n. For $g(x) \varepsilon C^{\infty}$ there is a positive number $M^* > 0$ such that $|g| \leq M^*$ for all $x \varepsilon [a,b]$. Therefore, $\{f_n g\}$ is boundedly convergent if $\{f_n\}$ is and $g \varepsilon C^{\infty}$.

Theorem VI

If $\{f_n(x)\}$ is boundedly convergent on the interval $[a, b]$, if each $f_n(x)$ is continuous on $[a, b]$, and if the limit $\lim_{n \to \infty} f_n(x) = f(x)$ is Riemann integrable on $[a, b]$, then with $t \varepsilon [a, b]$ the sequence of integrals,

$$\left\{ \int_a^t f_n \, dx \right\}$$

will be boundedly convergent on $[a, b]$ and each integral as well as the limit of the integrals will be Riemann integrable on $[a, b]$.

The Proof for Theorem VI is as follows.

Since $f_n(x)$ is continuous on $[a, b]$ for each n, $f_n(x)$ is Riemann integrable on $[a, b]$. For $f_n(x)$ Riemann integrable on

$[a, b]$, it follows that $\int_a^t f_n \, dx$ is Riemann integrable on $[a, b]$.

From Theorem IV,

$$\lim_{n \to \infty} \int_a^t f_n dx = \int_a^t f(x) \, dx$$

for each given point $t \in [a, b]$. The function $\int_a^t f \, dx$ is continuous on $[a, b]$ and hence it is Riemann integrable on $[a, b]$. This proves the last part of VI. To prove that the sequence of integrals is boundedly convergent on $[a, b]$ consider the pointwise convergence of this sequence. For each $t \in [a, b]$

$$\lim_{n \to \infty} \int_a^t f_n dx = \int_a^t f(x) \, dx$$

so that pointwise convergence holds. It is known that if the function $f_n(x)$ is continuous on $[a, t]$ then

$$\left| \int_a^t f_n(x) \, dx \right| \leq M \, (f_n) \cdot [t-a]$$

where $M(f_n)$ = the maximum of $|f_n(x)|$ on $[a, t]$.( )

Since $\{f_n\}$ is uniformly bound by a number $M > 0$ on $[a, b]$,

$$\left| \int_a^t f_n dx \right| \leq M \cdot [t - a]$$

for all $n = 1,2,3,\ldots$. Also, since $[t - a] \leq [b - a]$ for all $t \in [a, b]$ it follows that

$$\left| \int_a^t f_n dx \right| \leq M \cdot [b - a]$$

for all n and all $t \varepsilon [a, b]$. That is, $\left\{\int_a^t f_n dx\right\}$ is boundedly convergent on $[a, b]$.

APPENDIX III

IMPORTANT SEQUENCES IN $\bigcup_{t\epsilon T} \hat{g}_t$

Case (I)

Consider the normal function

$$S(x) = \begin{cases} \dfrac{\exp\left[\dfrac{-1}{1-x^2}\right]}{\int_{-\infty}^{\infty} \exp\left[\dfrac{-1}{1-x^2}\right] dx} , & |x| \leq 1 \\ 0 , & |x| \geq 1 \end{cases} \tag{A3-1}$$

It is seen that $S(x)$ has all orders of derivatives except possibly at $x = \pm 1$. But it is found that

$$D_x^{(i)} S(x)\Big|_{x=\pm 1} = 0 \quad \text{for any} \quad i = 1,2,\ldots .$$

Then $S(x)$ is in C^∞.

Some additional properties of $S(x)$ are

(i) $S(x)$ is greater than or equal to zero.

(ii) $S(x) = S(-x)$

(iii) $\int_{-1}^{+1} S(x)dx = 1.$

For $t_j \neq a_1$ a fixed number in E_1 consider the sequence $\{S_n(t, t_j)\}$ formed from $S(x)$ as follows.

For each $n = 1,2,\ldots$ let

$$S_n(t, t_j) = nS(nt-nt_j) = \begin{cases} n\cdot\exp\left[\dfrac{-1}{1-n^2(t-t_j)^2}\right], & |n(t-t_j)| \le 1 \\ 0, & |n(t-t_j)| > 1 \end{cases} \tag{A3-2}$$

For any given n, $S_n(t, t_j) \in C^{\infty}$. Also

$$\int_{t_j-\frac{1}{n}}^{t} S_n(x, t_j)ds = \begin{cases} 0, & t \le t_j - \frac{1}{n} \\ \text{Between } 0 \text{ and } +1, & t_j - \frac{1}{n} \le t \le t_j + \frac{1}{n} \\ 1, & t_j + \frac{1}{n} \le t \end{cases}$$

Then $\left\{f^1(S_n)\right\}$ is boundedly convergent and $\lim\limits_{n\to\infty} f^1(S_n)$ is Riemann integrable on each interval $[a_1, x_o]$ in E_1. The latter statement follows since if $a_1 < t_j$,

$$\lim_{n\to\infty}\int_{a_1}^{t} S_n\,dx = \left\{\begin{matrix} 1, & t > t_j \\ \frac{1}{2}, & t = t_j \\ 0, & t < t_j \end{matrix}\right]$$

and if $a_1 > t_j$,

$$\lim_{n\to\infty}\int_{a_1}^{t} S_n\,dx = \left\{\begin{matrix} 0, & t > t_j \\ -\frac{1}{2}, & t = t_j \\ -1, & t < t_j \end{matrix}\right]$$

For any positive integer i,

$$D_t^{(i)} S_n(t, t_j) \in C^{\infty}, \text{ and if } t \ne t_j \tag{A3-3}$$

$\lim\limits_{n\to\infty} D_t^{(i)} S_n = 0$; but for $t = t_j$ the limit may not exist.

For $k > 1$,

$$\lim_{n \to \infty} f_{x_1}^{k}(S_n) = f_{x_1}^{k-1}\left[\lim_{n \to \infty} f^{1}(S_n)\right] \tag{A3-4}$$

which exists for each x_1 in E_1. This relation follows from Theorem IV of Appendix II by noting the bounded convergence of $\left\{f^1(S_n)\right\}$ and the Riemann integrability of $\lim_{n \to \infty} f^1(S_n)$ on any given $[a_1, x_o]$ in E_1. Then for each integer K, $\lim_{n \to \infty} f_{x_1}^{k}(S_n)$ exists in E_1 for all but possibly a finite number of points $\neq a_1$ in any given $[a_1, x_o]$. Thus $\left\{S_n(t, t_j)\right\}$ satisfies equation (2-2) and is a member of S_{a_1}.

<u>Case II</u>

For each $n = 1,2,\ldots$ let

$$f_n(t, t_j) = \begin{cases} 0 \quad , & t < t_j - \frac{1}{2n} \\ n \quad , & t_j - \frac{1}{2n} \leq t \leq t_j + \frac{1}{2n} \\ 0 \quad , & t_j + \frac{1}{2n} < t \end{cases} \tag{A3-5}$$

with $t_j \neq a_1$ a fixed number in E_1. It can be seen that

$$\int_{t_j - \frac{1}{2n}}^{t} f_n(t, t_j)dx = \begin{cases} 0 & , \quad t \leq t_j - \frac{1}{2n} \\ \text{Between 0 and +1}, & t_j - \frac{1}{2n} \leq t \leq t_j + \frac{1}{2n} \\ 1 & , \quad t \geq t_j + \frac{1}{2n} \end{cases}$$

As in the case of $\left\{S_n(t, t_j)\right\}$ it is found that $\left\{f^1(f_n)\right\}$ is boundedly convergent and $\lim_{n \to \infty} f^1(f_n)$ is Riemann integrable on each $[a_1, x_o]$.

For $i = 1,2,\ldots$ and any $n = 1,2,\ldots$,

$$D_t^{(i)} f_n(t, t_j) = 0 \quad \text{for} \quad t \neq t_j \pm \frac{1}{2n}$$

For any given n, no derivative exists at $t_j \pm \frac{1}{2n}$ for any $m = 1,2,\ldots$.

For any t in E_1 and any $i \geq 1$, $\lim_{n\to\infty} D_t^{(i)}(f_n) = 0$.

For $k > 1$,

$$\lim_{n\to\infty} f_{x_1}^k(f_n) = f_{x_1}^{k-1}[\lim_{n\to\infty} f^1(f_n)] \tag{A3-6}$$

exists in E_1 for each x_1 in E_1. This relation follows from Theorem IV of Appendix II since $\{f^1(f_n)\}$ is boundedly convergent and $\lim_{n\to\infty} f^1(f_n)$ is Rie. Int. on each $[a_1, x_o]$.

It follows from these arguments that if k is any integer and $t_j \neq a_1$

$$\lim_{n\to\infty} \left| f_{x_1}^k(S_n) - f_{x_1}^k(f_n) \right| = 0 \tag{A3-7}$$

for all x_1 in E_1 except possibly a finite number of points different from a_1 in each $[a_1, x_o] \in E_1$.

That is $\{f_n\}$ is equivalent to $\{S_n\}$ where $\{S_n\}$ is a member of S_{a_1} while $\{f_n\}$ is only a member of $\mathcal{B}$.

Case III

Let b belong to B and suppose b is infinitely differentiable at a_1 where a_1 is defined in equation (2-1). In general, there may be a countably infinite number of isolated points in E_1 where b is not infinitely differentiable, but there can be only a finite number of such points in any finite interval of E_1. Two points t_j and t_{j+1} in E_1 are said to be isolated if there exists a number $\sigma > 0$ such that $|t_{j+1} - t_j| > \sigma$. The constant sequence $\{b_n\}$, where $b_n = b$, is a member of $\mathcal{B}$ by definition. It is desired to find a sequence $\{F_n\}$ in S_{a_1} that is equivalent to $\{b_n\}$, $b_n = b$. Consider the more general case where there are a non-finite number of isolated points in E_1 where b fails to have all orders of derivatives. These isolated points may be described by the countable set $\{t_j | j = 0, \pm 1, \pm 2, \ldots\}$ where $t_j \neq a_1$ for any j, and where $t_{j-1} < t_j < t_{j+1}$.

Let $\{F_n\}$ be constructed as follows. For any given $j = 0, \pm 1, \ldots,$ let

$$X_{j,m_j}(t) = \int_{t_j - \frac{m_j}{n}}^{t} S_{n,m_j}(x, t_j)dx \tag{A3-8}$$

where for each n,

$$S_{n,m_j}(t, t_j) = \begin{cases} \left(\frac{n}{m_j}\right) \cdot \dfrac{\exp\left[\dfrac{-1}{1-\left(\frac{n}{m_j}\right)^2 (t-t_j)^2}\right]}{\int_{-\infty}^{\infty} \exp\left[\dfrac{-1}{1-x^2}\right] dx}, & \text{if } \left|\frac{n}{m_j}(t-t_j)\right| \leq 1 \\ 0, & \text{if } \left|\frac{n}{m_j}(t-t_j)\right| \geq 1 \end{cases} \tag{A3-9}$$

Here m_j is defined to be the minimum of the two positive numbers $\frac{1}{2}(t_{j+1} - t_j)$ and $\frac{1}{2}(t_j - t_{j-1})$ for each $j = 0, \pm 1, \ldots$.

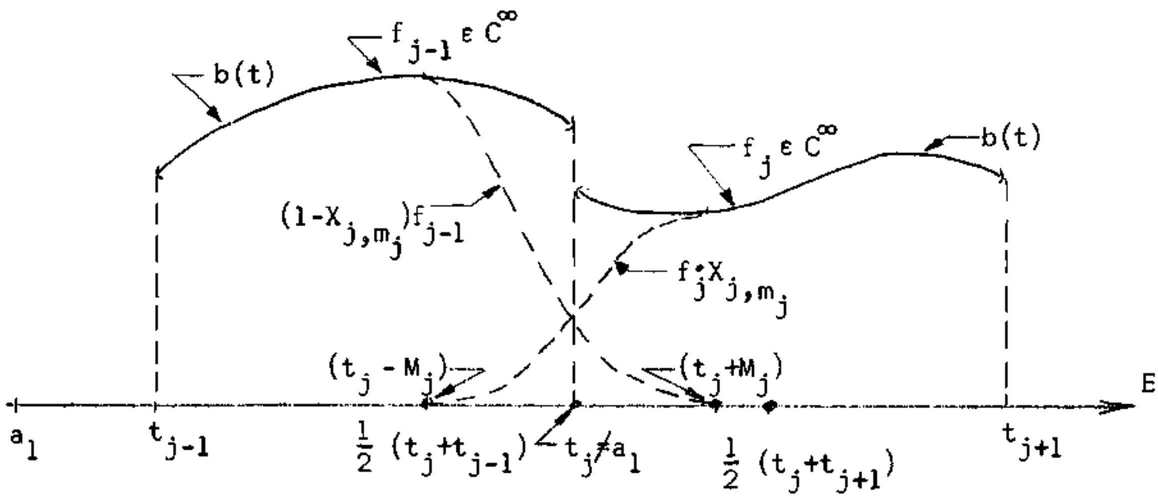

Figure 11. Construction of the Sequence of Normal Functions $\{F_n\}$.

For each $j = 0, \pm 1, \pm 2, \ldots$ let $f_j(t) \varepsilon C^{\infty}$ be a normal function equal to $b(t)$ for $t \varepsilon (t_j, t_{j+1})$.

Let

$$F_n(t) = f_{j-1}(t)\,[1 - X_{j,m_j}(t)] + f_j(t) \cdot X_{j,m_j}(t) \qquad \text{(A3-10)}$$

for all $t \varepsilon [t_j - m_j, t_j + m_j]$ where $j = 0, \pm 1, \ldots$ and let $F_n(t) = b(t)$ for each n when $t \varepsilon E_1 - \left\{[t_j - m_j, t_j + m_j] \mid j = 0, \pm 1, \ldots\right\}$. The notation $E_1 - \left\{[t_j - m_j, t_j + m_j] \mid j = 0, \pm 1, \ldots\right\}$ means the set of all points in E_1 not contained in the collection $\left\{[t_j - m_j, t_j + m_j] \mid j = 0, \pm 1, \ldots\right\}$. The construction of F_n is shown in Figure 11.

Since m_j is a positive number for each j, F_n has all orders of derivatives at every point t in $E_1 - \left\{[t_j - m_j, t_j + m_j] \mid j = 0, \pm 1, \ldots\right\}$. Also, since f_j and X_{j,m_j} are in C^{∞} for each $j = 0, \pm 1, \ldots$, F_n has all orders of derivatives at every $t \in (t_j - m_j, t_j + m_j)$ for each j. If $t \in (t_j - m_j, t_j + m_j)$ the $i^{\underline{th}}$ derivative of F_n is given by

$$D_t^{(i)} F_n = \sum_{r=o}^{i} D_t^{(i-r)} f_{j-1} \cdot D_t^{(r)} [1 - X_{j,m_j}] + \sum_{r=o}^{i} D_t^{(i-r)} f_j \cdot D_t^{(r)} X_{j,m_j} \quad \text{(A3-11)}$$

The right-hand derivative of order i of F_n evaluated at $t = t_j - m_j$ is denoted by

$$D_t^{(i)} F_n \Big|_{(t_j - m_j)^+}$$

The left-hand derivative of order i for F_n evaluated at $t = t_j + m_j$ is denoted by

$$D_t^{(i)} F_n \Big|_{(t_j + m_j)^-}$$

If $r = 1, 2, \ldots, i$;

$$D_t^{(r)} X_{j,m_j} \Big|_{t = (t_j \pm m_j)} = 0 .$$

Then

$$D_t^{(i)} F_n \Big|_{(t_j - m_j)^+} = D_t^{(i)} f_{j-1} \cdot [1 - X_{j,m_j}] \Big|_{(t_j - m_j)} + D_t^{(i)} f_j \cdot X_{j,m_j} \Big|_{(t_j - m_j)}$$

and

$$D_t^{(i)} F_n \Big|_{(t_j + m_j)^-} = D_t^{(i)} f_{j-1} \cdot [1 - X_{j,m_j}] \Big|_{(t_j + m_j)} + D_t^{(i)} f_j \cdot X_{j,m_j} \Big|_{(t_j + m_j)}$$

However,

$$X_{j,m_j} = \begin{cases} 0 & , \quad t \leq t_j - m_j \\ 1 & , \quad t \geq t_j + m_j \end{cases}$$

Then $D_t^{(i)} F_n \Big|_{(t_j - m_j)+} = D_t^{(i)} f_{j-1} \Big|_{(t_j - m_j)}$

and $D_t^{(i)} F_n \Big|_{(t_j + m_j)-} = D_t^{(i)} f_j \Big|_{(t_j + m_j)}$. $D_t^{(i)} f_{j-1} \Big|_{(t_j - m_j)}$ is also the left-hand derivative of order i at $(t_j - m_j)$ for F_n, and $D_t^{(i)} f_j \Big|_{(t_j + m_j)}$ is the right-hand derivative of order i at $(t_j + m_j)$ for F_n.

Since the above argument holds for each $j = 0, \pm 1, \ldots$ it must follow that $F_n \varepsilon C^\infty$ for every $n = 1, 2, \ldots$.

Membership of $\{F_n\}$ in S_{a_1} can be shown as follows. For each j, M_j is a positive number. Also,

$$\lim_{n \to \infty} X_{j,m_j} = \begin{cases} 1 & , \quad t > t_j \\ \frac{1}{2} & , \quad t = t_j \\ 0 & , \quad t < t_j \end{cases}$$

If $t \varepsilon [t_j - m_j, t_j + m_j]$, $F_n^{(i)}$ is given by (A3-11) and for all other t in E_1, $F_n^{(i)}$ is $b^{(i)}$ for each integer $i \geq 0$. Also $D_t^{(r)} X_{j,m_j} = D_t^{(r-1)} S_{n,m_j}$. If $r > 0$,

$$\lim_{n \to \infty} D_t^{(r-1)} S_{n,m_j} = \begin{cases} 0 & , \quad t \neq t_j \\ \text{Possibly Undefined} & , \quad t = t_j \end{cases}$$

Therefore, $\lim_{n\to\infty} F_n^{(i)}$ exists in E_1 for all but possibly a finite number of points not equal to a_1 in each finite interval of E_1. The points of E_1 where their limit might not exist are t_j where $j = 0, \pm 1, \pm 2, \ldots,$ but $t_j \neq a_1$ for any j.

For any j and n, X_{j,m_j} is a non-decreasing function on E_1 and varies from zero for $t \leq t_j - \frac{m_j}{n}$ to one for $t \geq t_j + \frac{m_j}{n}$. This implies that the sequence $\{F_n\}$ is uniformly bounded on every $[a_1, x_o]$ in E_1. Consideration of $\lim_{n\to\infty} X_{j,m_j}$ shows that $\lim_{n\to\infty} F_n(t) = b(t)$ for each $t \in E_1 - \{[t_j - m_j, t_j + m_j] | j = 0, \pm 1, \ldots\}$ and

$$\lim_{n\to\infty} F_n(t) = f_{j-1}(t)[1 - h(t-t_j)] + f_j(t)\cdot h(t-t_j) \qquad \text{(A3-12)}$$

for all $t \in [t_j - m_j, t_j + m_j]$ where $j = 0, \pm 1, \ldots$. Then $\{F_n\}$ is boundedly convergent and each F_n as well as $\lim_{n\to\infty} F_n$ are Riemann integrable on every interval $[a_1, x_o]$ in E_1. In light of Theorem IV of Appendix II, these conditions are sufficient to insure that $\lim_{n\to\infty} f^k(F_n) = f^k(\lim_{n\to\infty} F_n)$ for every given $k > 1$.

Therefore, $\lim_{n\to\infty} f^k_{x_1}(F_n)$ exists in E_1 for all $x_1 \in E_1$ except possibly $x_1 = t_j$ where $j = 0, \pm 1, \ldots$. That is,

$$\{F_n\} \in S_{a_1}$$

Equivalence of $\{F_n\}$ and $\{b_n\}$, where $b_n = b$, can be shown as follows. For each $k < 0$ and any $t \in [t_j - m_j, t_j + m_j]$,

$$\lim_{n\to\infty} F_n^{(-k)} = \sum_{r=0}^{(-k)} \left(f_{j-1}^{(-k-r)} \cdot \lim_{n\to\infty} D_t^{(r)}[1 - X_{j,m_j}] + f_j^{(-k-r)} \cdot \lim_{n\to\infty} D_t^{(r)} X_{j,m_j} \right)$$

$$= \begin{cases} f_{j-1}^{(-k)} \quad , \quad t \in [t_j - m_j, t_j) \\ \text{May be undefined if } t = t_j \\ f_j^{(-k)} \quad , \quad t \in (t_j, t_j + m_j] \end{cases}$$

That is, $\lim_{n\to\infty} F_n^{(-k)}(t) = b^{(-k)}(t)$ if $t \in [t_j - m_j, t_j + m_j]$, but where $t \neq t_j$.

Then if $k < 0$, $\lim_{n\to\infty} |f_{x_1}^k(F_n) - f_{x_1}^k(b_n)| = 0$ for all $x_1 \in E_1$ except possibly for $x_1 = t_j$ where $j = 0, \pm 1, \ldots$.

If $k \geq 0$ the condition $\lim_{n\to\infty} f^k(F_n) = f^k(\lim_{n\to\infty} F_n)$ implies $\lim_{n\to\infty} |f_{x_1}^k(F_n) - f_{x_1}^k(b_n)| = 0$ for all x_1 except possibly $x_1 = t_j$ where $j = 0, \pm 1, \ldots$. Thus $\{F_n\} \sim \{b_n\}$ where $\{F_n\} \in S_{a_1}$ and $\{b_n\}$, $b_n = b$, is a sequence of $\mathcal{B}$.

The sequence $\{F_n\}$ in S_{a_1} that is equivalent to the constant sequence formed from equation (3-1) is given by

$$F_n(t) = f_o[1 - X_{1,m}] + \ldots + f_j[X_{j,m} - X_{j+1,m}] + \ldots + f_k X_{k,m}$$

where $X_{j,m}$ is defined to be 1 if $j = 0$ and 0 if $j = k + 1$. The number $m > 0$ used here is the minimum of $1, t_2 - t_1, t_3 - t_2, \ldots, t_k - t_{k-1}$, where m is always defined to be 1 if there is only one isolated point to consider. If there are <u>no</u> isolated points to consider, b is in C^∞.

The sequence $\{F_n\}$ in S_{a_1} that is equivalent to the constant sequence formed from equation (3-2) is given by

$$F_n = f_o(t)[1 - X_{j,T/2}] + f_o(t-T)[X_{j,T/2}]$$

for all $t \varepsilon (t_j - T/2, t_j + T/2]$ and $F_n(t) = F_n(t \pm NT)$ for $N = 1,2,\ldots$; where $f_o(t) \varepsilon C^\infty$ is given in (3-2) and where $t_j \pm NT \neq a_1$ for any $N = 0, 1, 2,\ldots$.

In this second case, it is seen that F_n is a periodic function in C^∞ for each $n = 1,2,\ldots$.

APPENDIX IV A

REVIEW OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

Let $L_m(x) = 0$ be the linear homogeneous differential equation of order m so that

$$L_m(x) = x^{(m)} + p_1(t)\, x^{(m-1)} + p_2(t)\, x^{(m-2)} + \dots + p_m(t) = 0 \quad \text{(A4-1)}$$

where $p_j(t)$, $j = 1,\dots,m$ are all members of C^1 on interval I.

The linear system of equations[35] associated with $L_m(x) = 0$, $t \in I$, is the vector equation

$$\hat{x}' = A(t)\hat{x} \quad \text{where for } t \in I ,$$

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -p_m & -p_{m-1} & -p_{m-2} & -p_{m-3} & \dots & -p_1 \end{pmatrix}$$

If $\varphi_1, \dots, \varphi_m$ are m linearly independent solutions for $L_m(x) = 0$, then the matrix

$$\Phi(t) = \begin{pmatrix} \varphi_1 , & \varphi_2 & \dots & \varphi_m \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(m-1)} & \varphi_2^{(m-1)} & \dots & \varphi_m^{(m-1)} \end{pmatrix}$$

is a fundamental matrix solution for $\hat{x}' = A(t)\hat{x}$. That is $\Phi' = A\Phi$

where $p_j \in C^1$ with $t \in I$.

The determinant of the matrix Φ is called the "Wronskian" of $L_m(x) = 0$ with respect to $\varphi_1,\ldots,\varphi_m$; and it is denoted by $W(\varphi_1,\ldots,\varphi_m)$. The Wronskian is a function of t on I for fixed $\varphi_1,\ldots,\varphi_m$; and its value is denoted by $W(\varphi_1,\ldots,\varphi_m)(t)$.

From the fact that

$$|\Phi(t)| = |\Phi(\tau)| e^{\int_\tau^t [\text{trace } A(s)] ds}, \qquad \text{(A4-4)}$$

with $t \in I$ and τ fixed in I, for a linear system such as $\hat{x}' = A\hat{x}$; it follows that

$$W(\varphi_1,\ldots,\varphi_m)(t) = W(\varphi_1,\ldots,\varphi_m)(\tau) e^{-\int_\tau^t p_1(s) ds} \qquad \text{(A4-5)}$$

for $t, \tau \in I$.[36] This is the case since trace $A = -p_1$ for the matrix $A(t)$ considered here.

If $\varphi_1,\ldots,\varphi_m$ is a fundamental set for the homogeneous equation $L_m(x) = 0$ with $p_j \in C^1$ on interval I, then the solution $\Psi(t)$ of the non-homogeneous equation

$$L_m(x) = x^{(m)} + p_1(t) x^{(m-1)} + \ldots + p_m(t) x = b(t); \qquad \text{(A4-6)}$$

with $b(t) \in C^1$ on I, satisfying $\psi^{(i)}(\tau) = \xi_{i+1}, (\tau \in I, |\xi_{i+1}| < \infty)$ for each $i = 0,\ldots,m-1$; is given by

$$\Psi(t) = \Psi_h(t) + \sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_\tau^t \frac{W_{k_o}(\varphi_1,\ldots,\varphi_m)}{W(\varphi_1,\ldots,\varphi_m)} b(s) ds \qquad \text{(A4-7)}$$

where $\Psi_h(t)$ is that unique solution of $L_m(x) = 0$ for which $\Psi_h^{(i)}(\tau) = \xi_{i+1}$, $i = 0,\ldots,m-1$.[37] In the above equation, for $\Psi(t)$, W_{k_o} is the determinant obtained from W by replacing the $k_o^{\underline{th}}$ column by $(0,\ldots,0,1)$.

The Linear Equation of Order m with Constants Coefficient

Consider L_m with $p_1,\ldots,p_m$ constants. The interval I is now the entire real axis E_1.

For the constant coefficient case, the matrix A in $\hat{x}' = A\hat{x}$ is a constant matrix.

A fundamental set of solutions of $L_m(x) = 0$ can be exhibited, and the precise form of these functions depends on the characteristic polynomial $f(\lambda) = |\lambda E - A|$, where E is the unit matrix. The characteristic polynomial $f(\lambda) = \lambda^m + p_1\lambda^{m-1} + \ldots + p_m$ is obtained formally from $L_m(x)$ by changing $x^{(k)}$ to $\lambda^{(k)}$, for each $k = 0,\ldots,m$.

If $\lambda_1,\ldots,\lambda_s$ are the distinct roots of $f(\lambda)$ and if λ_i has multiplicity m_i, $(i = 1,\ldots,s)$, then a fundamental set for $L_m(x) = 0$ is given by the m functions

$$t^j e^{t\lambda_i}, \quad (j = 0,\ldots,m_i - 1, \quad i = 1,\ldots, s)^{(38)} \qquad \text{(A4-8)}$$

Let $\varphi_1,\ldots,\varphi_m$ denote the m linearly independent solutions of $L_m(x) = 0$ of the form $t^j e^{t\lambda_i}$. It follows that the solution $\Psi(t)$ of $L_m(x) = b(t)$, where $b(t) \in C^1$ on E_1, satisfying $\Psi^i(\tau) = \xi_{i+1}$, $i = 0,\ldots,m-1$; is given by

$$\Psi(t) = \Psi_h(t) + \sum_{k_o=1}^{m} \varphi_{k_o} \int_\tau^t \left[\frac{e^{p_1 S}\, W_{k_o}(\varphi_1,\ldots,\varphi_m)(S)b(S)}{e^{p_1\tau}\, W(\varphi_1,\ldots,\varphi_m)(\tau)}\right] dS \qquad \text{(A4-9)}$$

where, as before, $\Psi_h(t)$ is that solution of $L_m(x) = 0$ for which $\Psi_h^{(i)}(\tau) = \xi_{i+1}$, $i = 0,\ldots,m-1$.

APPENDIX IV B

THE UNIQUENESS OF THE GENERALIZED SOLUTION OF $L_m(x) = g_t$

For $\{b_n\} \sim \{b_n^*\}$ in S_{a_1} it is possible to show the equivalence in S_{a_1} of the sequences

$$\left\{ \sum_{k_o=1}^{m} \varphi_{k_o} \int_{a_1}^{t} q_{k_o} \; b_n \; ds \right\}$$

and

$$\left\{ \sum_{k_o=1}^{m} \varphi_{k_o} \int_{a_1}^{t} q_{k_o} \; b_n^* \; ds \right\}$$

where q_{k_o} is used in place of $\frac{W_{k_o}}{W}$.

Since φ_{k_o}, q_{k_o} are complex in general,

$$\int_{a_1}^{t} q_{k_o} b_n ds = \int_{a_1}^{t} \mathrm{Re}(q_{k_o}) \cdot b_n ds + j \int_{a_1}^{t} \mathrm{Im}(q_{k_o}) \cdot b_n ds$$

Let
$$J_{k_o} = \varphi_{k_o} \cdot \int_{a_1}^{t} q_{k_o} b_n \; ds$$

$$= \mathrm{Re}\, \varphi_{k_o} \cdot \int_{a_1}^{t} \mathrm{Re}(q_{k_o}) \cdot b_n \; ds$$

$$- \mathrm{Im}\, \varphi_{k_o} \cdot \int_{a_1}^{t} \mathrm{Im}(q_{k_o}) \cdot b_n \; ds$$

$$+ j \cdot \mathrm{Re}\, \varphi_{k_o} \cdot \int_{a_1}^{t} \mathrm{Im}(q_{k_o}) \cdot b_n ds$$

$$+ j\, \mathrm{Im}\, \varphi_{k_o} \cdot \int_{a_1}^{t} \mathrm{Re}(q_{k_o}) \cdot b_n \; ds$$

Then

$$\sum_{k_o=1}^{m} \varphi_{k_o} \int_{a_1}^{t} q_{k_o} b_n \, ds = \sum_{k_o=1}^{m} \mathrm{Re}(J_{k_o})$$

$$+ j \sum_{k_o=1}^{m} \mathrm{Im}\,(J_{k_o}) .$$

The solution of $L_m(x) = b_n$ with $\psi_n^{(i)}(a_1) = 0$, for $i = 0,\ldots,m-1$, is

$$\psi_n(t) = \sum_{k_o=1}^{m} \varphi_{k_o} \cdot \int_{a_1}^{t} q_{k_o} \, b_n ds$$

Suppose ψ_n is not a real function. Then $L_m(\psi_n) = L_m(\mathrm{Re}\psi_n) + jL_m(\mathrm{Im}\psi_n) = b_n$ where b_n is known to be real.

Then $L_m(\mathrm{Re}\psi_n) = b_n$ and $L_m(\mathrm{Im}\psi_n) = 0$.

The initial conditions give $\mathrm{Re}\psi_n^{(i)}(a_1) = 0$ and $\mathrm{Im}\,\psi_n^{(i)}(a_1) = 0$, for $i = 0,\ldots,m-1$.

However the solution of $L_m[\mathrm{Im}\,\psi_n] = 0$ with $\mathrm{Im}\,\psi_n^{(i)}(a_1) = 0$ is the zero solution. That is for the given initial conditions, $\mathrm{Im}\,\psi_n = 0$, and hence ψ_n is a real function.

Then $\sum_{k_o=1}^{m} \mathrm{Im}(J_{k_o}) = 0$.

The real and imaginary parts of both φ_{k_o} and q_{k_o} are members of C^∞. Therefore, if $\{b_n\} \sim \{b_n^*\}$ in S_{a_1},

$$\left\{\sum_{k_o=1}^{m} \varphi_{k_o} \cdot \int_{a_1}^{t} q_{k_o} b_n ds\right\} \sim \left\{\sum_{k_o=1}^{m} \varphi_{k_o} \cdot \int_{a_1}^{t} q_{k_o} b_n^* \, ds\right\} \text{ in } S_{a_1} .$$

If $\{b_n\} \sim \{b_n^*\}$, the sequences of solutions $\{\psi_n\}$ and $\{\psi_n^*\}$ of $L_m(x) = b_n$ and $L_m(x) = b_n^*$, respectively, must be equivalent sequences in S_{a_1}.

APPENDIX V

SOLUTIONS OF $L_m(x) = g_t$ FOR SOME IMPORTANT FUNCTIONS IN G

Type 1

Consider the sequence $\{S_n\}$ of S_{a_1} defined in equation (3-4). Substituting members of $\{S_n\}$ into equation (6-1) and letting $q_{k_o} = \frac{W_{k_o}}{W}$, the sequence of particular solutions of $L_m(x) = S_n$, $n = 1,2,\ldots,$ are given by

$$\Psi_{n,p} = \sum_{k_o=1}^{m} \varphi_{k_o} \int_{a_1}^{t} q_{k_o} S_n \, ds \tag{A5-1}$$

If $q_1 < t_j$,

$$\lim_{n\to\infty} \Psi_{n,p} = \begin{cases} 0 & , \quad t < t_j \\ \frac{1}{2} \sum\limits_{k_o=1}^{m} \varphi_{k_o}(t_j)\cdot q_{k_o}(t_j) & , \quad t = t_j \\ \sum\limits_{k_o=1}^{m} \varphi_{k_o}(t)\cdot q_{k_o}(t_j) & , \quad t > t_j \end{cases} \tag{A5-2}$$

The constant sequence $\left\{\lim\limits_{n\to\infty} \Psi_{n,p}\right\}$ is in $\mathcal{B}$. Since $\varphi_{k_o} \in C^\infty$ for each $k_o = 1,2,\ldots,m,$

$$f^{k}[\lim_{n\to\infty}\Psi_{n,p}] = \begin{cases} 0 & , \quad t < t_j \\ 0 \text{ when the derivative is undefined at } t_j & \\ \sum_{k_o=1}^{m} \varphi_{k_o}^{(-k)} \cdot q_{k_o}(t_j) & , \quad t > t_j \end{cases} \tag{A5-3}$$

when $k < 0$ is considered.

If $i = 1,2,\ldots,$

$$D_t^{(i)}\Psi_{n,p} = \sum_{k_o=1}^{m}\sum_{r=0}^{i} A_r D_t^{(i-r)}(\varphi_{k_o}) \cdot D_t^{(r-1)}(q_{k_o} \cdot S_n)$$

where $D_t^{-1}(q_{k_o} S_n)$ is defined to be $\int_{a_1}^{t} q_{k_o} S_n \, ds$. Each A_r is a constant with $A_o = A_m = 1$.

If $i \geq 0$, $\lim_{n\to\infty} S_n^{(i)} = 0$ for any $t \neq t_j$.

Then $\lim_{n\to\infty} D_t^{(i-r)}(\varphi_{k_o}) \cdot D_t^{(r-1)}(q_{k_o} \cdot S_n) = 0$ for each t in E_1 except possibly $t = t_j$ where $r = 1,2,\ldots, i$.

For $r = 0$,

$$\lim_{n\to\infty} D_t^{(i-r)}(\varphi_{k_c}) \cdot D_t^{(r-1)}(q_{k_o} S_n) = \begin{cases} 0 & , \quad t < t_j \\ \frac{1}{2}\varphi_{k_o}^{(i)}(t_j) \cdot q_{k_o}(t_j), & \quad t = t_j \\ \varphi_{k_o}^{(i)}(t) \cdot q_{k_o}(t_j) & , \quad t > t_j \end{cases}$$

Then if $k < 0$,

$$\lim_{n\to\infty} f^k(\psi_{n,p}) = \begin{cases} 0 & , \quad t < t_j \\ \text{Possibly Undefined} & , \quad t = t_j \\ \varphi_{k_o}^{(-k)}(t) \cdot q_{k_o}(t_j) & , \quad t > t_j \end{cases} \tag{A5-4}$$

$\lim\limits_{n\to\infty} f^k(\psi_{n,p})$ is now investigated for integers $k > 0$.

Consider the sequence $\left\{\int_{a_1}^{t} q_{k_o} S_n \, ds\right\}$.

From Theorems II and III of Appendix II,

$$\int_{a_1}^{t} q_{k_o} \cdot S_n \, ds = \int_{a_1}^{t} q_{k_o} \, d\left[\int_{a_1}^{s} S_n \, dx\right].$$

The following theorem[34] is to be used for the above expression.

If a function $f(x)$ is continuous on $[a, b] \varepsilon E_1$ and if $g(x)$ has finite variation on $[a, b]$,

$$\left|\int_a^b f(x) \, d[g(x)]\right| \leq M(f) \cdot \overset{b}{\underset{a}{V}} (g)$$

where $\overset{b}{\underset{a}{V}} (g)$ is the total variation of $g(x)$ on $[a, b]$ and $M(x)$ is the maximum of $|f(x)|$ on $[a, b]$. $\int_{a_1}^{s} S_n \, dx$ is a monotonic function on any given $[a_1, x_o]$ in E_1.

For any n, the total variation of $\int_{a_1}^{s} S_n \, dx$ on any given $[a_1, x_o]$ can not exceed unity, from the nature of S_n. Since $g_{k_o} \varepsilon C^{\infty}$ the maximum of $|g_{k_o}|$ is bounded above on any interval $[a_1, x_o]$ in E_1.

With $M(g_{k_o})$ the maximum of $|q_{k_o}|$ on $[a_1, x_o]$,
$\left|\int_{a_1}^{t} g_{k_o} S_n \, ds\right| \leq M(g_{k_o})$ for all n and all $t \varepsilon [a_1, x_o]$.

Then $\left\{\int_{a_1}^{t} q_{k_o} S_n \, ds\right\}$ is uniformly bounded on each $[a_1, x_o]$ in E_1.

For $a_1 < t_j$,

$$\lim_{n\to\infty} \int_{a_1}^{t} q_{k_o} S_n \, ds = \begin{cases} 0 & , \quad t < t_j \\ \frac{1}{2} q_{k_o}(t_j) & , \quad t = t_j \\ q_{k_o}(t) & , \quad t > t_j \end{cases}$$

Therefore, $\left\{\int_{a_1}^{t} q_{k_o} S_n \, ds\right\}$ is boundedly convergent and $\lim_{n\to\infty} \int_{a_1}^{t} q_{k_o} S_n \, ds$ is Riemann integrable on $[a_1, x_o] \, \varepsilon \, E_1$.

With Theorems V and VI of Appendix II and the above results on bounded convergence, it follows that

$$\lim_{n\to\infty} f^k(\Psi_{n,p}) = f^k(\lim_{n\to\infty} \Psi_{n,p}) \quad \text{where } k > 0. \tag{A5-5}$$

For every integer k found with reference to equations (A5-3), (A5-4) and (A5-5), it is found that

$$\lim_{n\to\infty} \left| f^k_{x_1}(\Psi_{n,p}) - f^k_{x_1}(\lim_{n\to\infty} \Psi_{n,p}) \right| = 0 \tag{A5-6}$$

for all x_1 in E_1 except possibly $x_1 = t_j$. Then $\{\Psi_{n,p}\} \sim \{\lim_{n\to\infty} \Psi_{n,p}\}$ where $\{\Psi_{n,p}\}$ is in S_{a_1} and is the sequence of particular solutions of $L_m(x) = S_n$ where $\{S_n\}$ is the sequence (3-4).

<u>Sequence Type 2</u>

For any $i \geq 1$ the sequence of derivatives $\left\{S_n^{(i)}\right\}$ is in S_{a_1}

since $\left\{S_n\right\} \varepsilon S_{a_1}$.

Substituting $\left\{S_n^{(i)}\right\}$ in (6-1)

$$\Psi_{n,p} = \sum_{k_o=1}^{m} \varphi_{k_o}(t) \int_{a_1}^{t} q_{k_o} S_n^{(i)} ds \tag{A5-7}$$

Using integration by parts,

$$\int_{a_1}^{t} q_{k_o} S_n^{(i)} ds = q_{k_o}(s) \cdot \int_{a_1}^{s} S_n^{(i)} dx \Bigg|_{a_1}^{t} - \int_{a_1}^{t} q_{k_o}^{(1)} \cdot \int_{a_1}^{s} S_n^{(i)} dx \cdot ds \tag{A5-8}$$

In a similar way

$$\int_{a_1}^{t} q_{k_o}^{(r)} \cdot S_n^{(i-r)} ds = q_{k_o}^{(r)} \cdot S_n^{(i-r-1)} - q_{k_o}^{(r)} \cdot S_n^{(i-r-1)} \Bigg|_{t=a_1} \tag{A5-9}$$

$$- \int_{a_1}^{t} q_{k_o}^{(r+1)} \cdot S_n^{(i-r-1)} ds \quad \text{for} \quad r = 0,1,2,\ldots,m-1 \text{ .}$$

Expanding (A5-8) with (A5-9) until $\int_{a_1}^{t} q_{k_o}^{(i)} S_n ds$ is the last term for the expansion there is obtained

$$\int_{a_1}^{t} q_{k_o} S_n^{(i)} ds = \sum_{r=o}^{i=1} (-1)^r q_{k_o}^{(r)} S_n^{(i-r-1)} \tag{A5-10}$$

$$+ \sum_{r=o}^{i=1} (-1)^{r+1} q_{k_o}^{(r)} S_n^{(i-r-1)}$$

$$+ (-1)^i \int_{a_1}^{t} q_{k_o}^{(i)} S_n ds$$

Then with $a_1 < t_j$,

$$\lim_{n\to\infty} \psi_{n,p} = \begin{cases} 0 & , \quad t < t_j \\ \text{Possibly Undefined} & , \quad t = t_j \\ \sum_{k_o=1}^{m} (-1)^i \varphi_{k_o}(t)\, q_{k_o}^{(i)}(t_j) & , \quad t > t_j \end{cases} \tag{A5-11}$$

Expression (A5-10) can be used to expand $\psi_{n,p}$ given in (A5-7).

In general both φ_{k_o} and $q_{k_o}^{(r)}$, $r = 0,\ldots,m-1$, may be complex functions. For each n, $\psi_{n,p}$ will be real however. The expanded form of (A5-7) expressed in terms of the real and imaginary parts of φ_{k_o} and q_{k_o} is

$$\begin{aligned} \psi_{n,p} = \sum_{k_o=1}^{m} \Bigg[& \sum_{r=o}^{i-1} (-1)^r \cdot \mathrm{Re}\,(\varphi_{k_o}(t) \cdot q_{k_o}^{(r)}(t)) \cdot S_n^{(i-r-1)}(t,t_j) \\ & + \sum_{r=o}^{i-1} (-1)^{r+1} \cdot \mathrm{Re}(\varphi_{k_o}(t) \cdot q_{k_o}^{(r)}(a_1)) \cdot S_n^{(i-r-1)}(a_1,t_j) \\ & + (-1)^i \int_{a_1}^{t} \mathrm{Re}(\varphi_{k_o}(t) \cdot q_{k_o}^{(i)}(s)) \cdot S_n(s,t_j)\,ds \Bigg] \end{aligned} \tag{A5-12}$$

Since Real and Imaginary parts of φ_{k_o} and $\varphi_{k_o}^{(r)}$ are members of C^∞, and since $\mathrm{Re}(ZY) = \mathrm{Re}Z \cdot \mathrm{Re}Y - \mathrm{Im}Z \cdot \mathrm{Im}Y$ for two complex functions Z and Y, the sequence $\left\{\mathrm{Re}(\varphi_{k_o}(t) \cdot q_{k_o}^{(r)}(t)) \cdot S_n^{(i-r-1)}(t,t_j)\right\}$ is in S_{a_1}. Also, $\left\{\mathrm{Re}(\varphi_{k_o} \cdot q_{k_o}^{(r)}(a_1)) \cdot S_n^{(i-r-1)}(a_1,t_j)\right\}$ is in S_{a_1} and is equivalent to the constant sequence $\{0\}$, since $q_1 \neq t_j$ implies $\lim_{n\to\infty} S_n^{(i-r-1)}(a_1,t_j) = 0$. Finally, the sequence $\left\{\int_{a_1}^{t} \mathrm{Re}(\varphi_{k_o}(t) \cdot q_{k_o}^{(i)}(s)) \cdot S_n(s,t_j)\,ds\right\}$ is a member of

S_{a_1} and is equivalent to the constant sequence $\left\{H_{k_o}^{(i)}\right\}$ where

$$H_{k_o}^{(i)} = \begin{cases} 0 & , \quad t < t_j \\ \frac{1}{2} \operatorname{Re} \left(\varphi_{k_o}(t_j) \cdot q_{k_o}^{(i)}(t_j)\right) & , \quad t = t_j \\ \operatorname{Re}\left(\varphi_{k_o}(t) \cdot q_{k_o}^{(i)}(t_j)\right) & , \quad t > t_j \end{cases} \tag{A5-13}$$

It is noted that $\left\{H_{k_o}^{(i)}\right\}$ is in $\mathcal{B}$ but is not necessarily a member of S_{a_1}.

The sequence of solutions in (A5-7) is a sequence in S_{a_1} and is equivalent to the sum of the two sequences

$$\left\{ \sum_{k_o=1}^{m} \sum_{r=o}^{i-1} (-1)^r \operatorname{Re}(\varphi_{k_o} \cdot q_{k_o}^{(r)}) \cdot S_n^{(i-r-1)} \right\}$$

and

$$\left\{ \sum_{k_o=1}^{m} (-1)^i \, H_{k_o}^{(i)} \right\}.$$

The second of the sequences is a constant sequence of $\mathcal{B}$ while the first sequence is a composite of sequences of the form $\left\{S_n^{(r)}\right\}$ where $r = 0,\ldots,i-1$.

It can be seen that the limit of $\Psi_{n,p}$ as $n \to \infty$ may not exist for an excitation sequence $\left\{S_n^{(i)}\right\}$ where $i \geq 1$. Even if this limit does exist it does not necessarily follow that the sequences $\left\{\Psi_{n,p}\right\}$ and $\left\{\lim_{n \to \infty} \Psi_{n,p}\right\}$ are equivalent.

Sequence Type 3

Let $g_t \in G$ and suppose $\left\{b_n\right\} \in g_t$ where $\left\{b_n\right\}$ has the following properties:

Let $\lim_{n \to \infty} b_n = b$, with b infinitely differentiable at a_1 belong to B.

Let $\{b_n\}$ be equivalent to the constant sequence $\{b\}$.

Also, let $\{b_n\}$ be boundedly convergent on each $[a_1, x_o]$ in E_1.

From Theorems IV, V, and VI of Appendix II,

$$\lim_{n \to \infty} \int_{a_1}^{t} (q_{k_o} \circ b_n) ds = \int_{a_1}^{t} (q_{k_o} \cdot b) ds \tag{A5-14}$$

Then with $\{b_n\}$ substituted in (6-1)

$$\lim_{n \to \infty} \psi_{n,p} = \sum_{k_o=1}^{m} \varphi_{k_o} \circ \int_{a_1}^{t} (q_{k_o} \cdot b) ds \tag{A5-15}$$

It is easily seen that the constant sequence $\left\{\lim_{n \to \infty} \psi_{n,p}\right\}$ is in $\mathcal{B}$.

For $k < 0$, $f^k(\varphi_{k_o} \cdot \int_{a_1}^{t} (q_{k_o} \circ b_n) ds) = \sum_{r=o}^{(-k)} A_r \, D_t^{(-k-r)}(\varphi_{k_o}) \cdot D_t^{(r)} \int_{a_1}^{t} (q_{k_o} \circ b_n) ds$, where each A_r is an expansion constant with $A_o = A_{-k} = 1$.

For each $r = 1, \ldots, (-k)$; $D_t^{(r)} \int_{a_1}^{t} (q_{k_o} b_n) ds = D_t^{(r-1)}(q_{k_o} b_n)$ since $q_{k_o} b_n$ has all orders of derivatives.

Then for each $r = 1, \ldots, (-k)$

$$D_t^{(r)} \int_{a_1}^{t} (q_{k_o} b_n) ds = \sum_{p=o}^{r-1} A_p \, D_t^{(r-1-p)}(q_{k_o}) \cdot D_t^{(p)}(b_n)$$

where A_p are expansion constants with $A_{p=o} = A_{p=r-1} = 1$.

Then if $k < 0$,

$$f^k\left(\sum_{k_o=1}^{m} \varphi_{k_o} \cdot \int_{a_1}^{t} (q_{k_o} \cdot b_n)ds\right) = \sum_{k_o=1}^{m} \left\langle D_t^{(-k)}(\varphi_{k_o}) \cdot \int_{a_1}^{x} (q_{k_o} b_n)ds + \right.$$

$$\left. \sum_{r=o}^{-k} \left[A_r D_t^{(-k-r)}(\varphi_{k_o}) \cdot \left(\sum_{p=o}^{r-1} A_p D_t^{(r-1-p)}(q_{k_o}) \cdot D_{x_1}^{(p)}(b_n)\right)\right]\right\rangle \quad \text{(A5-16)}$$

For $k < 0$, $\sum_{k_o=1}^{m} f^k(\varphi_{k_o} \cdot \int_{a_1}^{t} (q_{k_o} \cdot b)ds)$ is given by (A5-16) if b_n is replaced with b. This is correct with the exception of possibly a finite number of points not equal to a_1 in each $[a_1, x_o]$ in E_1.

For $k < 0$, equation (A5-16) implies

$$\lim_{n\to\infty} \left| f_{x_1}^k (\Psi_{n,p}) - f_{x_1}^k (\lim_{n\to\infty} \Psi_{n,p})\right| = 0$$

for all x_1 except possibly a finite number of points different from a_1 in each $[a_1, x_o] \varepsilon E_1$.

With $\{b_n\}$ boundedly convergent, Theorems V and VI of Appendix II imply that

$$\{\Psi_{n,p}\} = \left\{\sum_{k_o=1}^{m} \varphi_{k_o} \cdot \int_{a_1}^{t} (q_{k_o} b_n) \, ds\right\}$$

is also boundedly convergent on each $[a_1, x_o]$ in E_1, since φ_{k_o} and q_{k_o} have all orders of derivatives.

$\Psi_{n,p}$ and $\lim_{n\to\infty} \Psi_{n,p}$ are Riemann integrable on each $[a_1, x_o]$. Then from Theorem IV of Appendix II,

$$\lim_{n\to\infty} f^k(\Psi_{n,p}) = f^k(\lim_{n\to\infty} \Psi_{n,p})$$

for each $k > 0$. But this means

$$\lim_{n \to \infty} f^k_{x_1} (\psi_{n,p}) - f^k_{x_1} (\lim_{n \to \infty} \psi_{n,p}) = 0$$

for x_1 in E_1.

Then $\{\psi_{n,p}\} \varepsilon S_{a_1}$, $\left\{\lim_{n \to \infty} \psi_{n,p}\right\} \varepsilon \mathcal{B}$ and the sequences are equivalent.

Then the particular solution of $L_m(x) = g_t$, where $\{b_n\} \varepsilon g_t$ is of type 3, is the member of G containing $\{\psi_{n,p}\}$, and embedded in this particular solution is the normal function $\lim_{n \to \infty} \psi_{n,p}$.

In the analysis of sequence types 1 and 3 the functions φ_{k_o} and q_{k_o}, $k_o = 1,\ldots,m$ have been operated on as though they were real functions. But φ_{k_o} and q_{k_o} are complex in general as mentioned in the development for sequence type 2. However, the analysis is the same for sequences of type 1 or 3 whether φ_{k_o} and φ_{k_o} are considered to be real or complex.

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